Ficha 4 - Trigonometria

① $\lim_{x\to 0} \frac{1}{x^2} \sin^2\left(\frac{x}{2}\right) \overset{0\times\infty}{=} \lim_{x\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{x} \times \frac{\sin\left(\frac{x}{2}\right)}{x}\right) = \lim_{y\to 0}\left[\left(\frac{\sin y}{2y}\right) \times \left(\frac{\sin y}{2y}\right)\right]$

$\frac{x}{2} = y$

$x = 2y$

$y \to 0$

$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ ⓒ

② $f(x) = 4\cos(2x)$.

2.1) $A = ?$

$f(x) = 0 \Leftrightarrow$

$\Leftrightarrow 4\cos(2x) = 0$

$\Leftrightarrow \cos(2x) = 0$

$\Leftrightarrow 2x = \frac{\pi}{2} + 2k\pi \vee 2x = -\frac{\pi}{2} + 2k\pi ; k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{4} + k\pi \vee x = -\frac{\pi}{4} + k\pi ; k \in \mathbb{Z}$.

$k = 0 \leadsto x = \frac{\pi}{4} \vee x = -\frac{\pi}{4}$

$k = 1 \leadsto x = \frac{5\pi}{4} \vee x = \frac{3\pi}{4}$

$k = -1 \leadsto x = -\frac{3\pi}{4} \vee x = -\frac{5\pi}{4}$

Assim, $A\left(\frac{\pi}{4} ; 0\right)$

$C = ?$

$f\left(-\frac{\pi}{6}\right) = 4\cos\left(2\times\left(-\frac{\pi}{6}\right)\right) =$

$= 4\cos\left(-\frac{\pi}{3}\right) =$

$= 4 \times \frac{1}{2}$

$= 2$

Assim $C\left(-\frac{\pi}{6} ; 2\right)$

$A_{\text{trapézio}} = \frac{B + b}{2} \times h =$

$= \frac{\frac{5\pi}{12} + \frac{\pi}{6}}{2} \times 2$

$= \frac{7\pi}{12}$

$B = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

$b = \frac{\pi}{6}$

$h = 2$

2.2) $f(x) = 4\cos(2x)$

$f'(x) = -4 \times 2\sin(2x) = -8\sin(2x)$

$f''(x) = -8 \times 2\cos(2x) = -16\cos(2x)$

$f(x) + f'(x) + f''(x) = 4\cos(2x) - 8\sin(2x) - 16\cos(2x) = -12\cos(2x) - 8\sin(2x)$

$= -4(3\cos(2x) + 2\sin(2x))$ c. q. d.

③ $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = g(0)$?

$g(0) = \ln k$

$\lim_{x\to 0^-} \ln(k - x) = \ln k$

$\lim_{x\to 0^+} \frac{\sin x}{3x} = \lim_{x\to 0^+}\left(\frac{1}{3} \times \underbrace{\frac{\sin x}{x}}_{1}\right) = \frac{1}{3} \times 1 = \frac{1}{3}$

$\ln k = \frac{1}{3}$

$k = e^{\frac{1}{3}}$

$k = \sqrt[3]{e}$

Ⓐ

④ $\sin\theta = \frac{\overline{OE}}{1} \Leftrightarrow \boxed{\overline{OE} = \sin\theta}$

$\cos\theta = \frac{\overline{EA}}{1} \Leftrightarrow \boxed{\overline{EA} = \cos\theta}$

$\boxed{\overline{OA} = 1}$

$P = (\sin\theta + \cos\theta + 1) \times 2$ ⓒ

(5) Seja $B(x,y)$

Então $f'(x) = 8$

$f'(x) = 2e^{2x} - \sin x - 4x$

R: $x \approx 0,91$

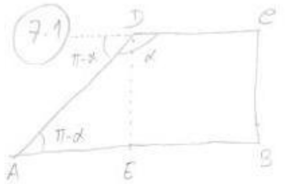
(6) $f(x) = a\cos(mx) + b\sin(mx)$

$f'(x) = -am\sin(mx) + bm\cos(mx)$

$f''(x) = -am^2\cos(mx) - bm^2\sin(mx)$

Ora $f''(x) + m^2 f(x) = -am^2\cancel{\cos(mx)} - bm^2\cancel{\sin(mx)} + m^2 a\cancel{\cos(mx)} + m^2 b\cancel{\sin(mx)} = 0$

(7.1)



$\sin(\pi-\alpha) = \frac{1}{\overline{AD}} \Leftrightarrow \sin\alpha = \frac{1}{\overline{AD}} \Leftrightarrow \boxed{\overline{AD} = \frac{1}{\sin\alpha}}$

$\cos(\pi-\alpha) = \frac{\overline{AE}}{\overline{AD}} \Leftrightarrow -\cos\alpha = \frac{\overline{AE}}{\frac{1}{\sin\alpha}} \Leftrightarrow \boxed{\overline{AE} = -\frac{\cos\alpha}{\sin\alpha}}$

Perímetro $= \underbrace{\overline{EB}}_{1} + \underbrace{\overline{BC}}_{1} + \underbrace{\overline{CD}}_{1} + \underbrace{\overline{DA}}_{\frac{1}{\sin\alpha}} + \underbrace{\overline{AE}}_{-\frac{\cos\alpha}{\sin\alpha}} = 3 + \frac{1}{\sin\alpha} - \frac{\cos\alpha}{\sin\alpha} = 3 + \frac{1-\cos\alpha}{\sin\alpha}$ c.q.d.

(7.2) $P'(\theta) = 3' + \left(\frac{1-\cos\theta}{\sin\theta}\right)' = \frac{(1-\cos\theta)'\sin\theta - (1-\cos\theta)(\sin\theta)'}{\sin^2\theta} =$

$= \frac{\sin\theta\,\sin\theta - (1-\cos\theta)\cos\theta}{\sin^2\theta} = \frac{\sin^2\theta - \cos\theta + \cos^2\theta}{\sin^2\theta}$

$= \frac{\overbrace{\sin^2\theta + \cos^2\theta}^{1} - \cos\theta}{\sin^2\theta} = \frac{1-\cos\theta}{\sin^2\theta}$

Ora $1 + tg^2\theta = \frac{1}{\cos^2\theta}$

$1 + (-\sqrt{8})^2 = \frac{1}{\cos^2\theta}$

$9 = \frac{1}{\cos^2\theta}$

$\cos\theta = \pm\frac{1}{3}$

$\cos\theta = -\frac{1}{3}$ $(\alpha \in]\frac{\pi}{2}, \pi[)$

$tg\,\theta = \frac{\sin\theta}{\cos\theta}$

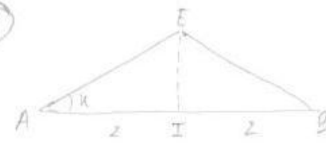
$-\sqrt{8} = \frac{\sin\theta}{-\frac{1}{3}}$

$\sin\theta = \frac{\sqrt{8}}{3}$

$P'(\theta) = \frac{1-\left(-\frac{1}{3}\right)}{\left(\frac{\sqrt{8}}{3}\right)^2} = \frac{\frac{4}{3}}{\frac{8}{9}} =$

$= \frac{3}{2}$

(8.1)



$tg\,x = \frac{\overline{EI}}{2}$

$\overline{EI} = 2\,tg\,x$

$A_{\triangle} = \frac{4 \times 2\,tg\,x}{2} = 4\,tg\,x$ e $A_{\square} = 4\times4 = 16$

Assim,

$A_{sombreada} = 4\times4 - 4\times4\,tg\,x =$
$= 16 - 16\,tg\,x =$
$= 16(1 - tg\,x)$

(8.2) $a(x)$ é cont. em $\left[\frac{\pi}{12}, \frac{\pi}{5}\right]$

$a\left(\frac{\pi}{12}\right) = 16\left(1 - tg\left(\frac{\pi}{12}\right)\right) \approx 11{,}71$

$a\left(\frac{\pi}{5}\right) = 16\left(1 - tg\left(\frac{\pi}{5}\right)\right) \approx 4{,}38$

$a\left(\frac{\pi}{5}\right) < 5 < a\left(\frac{\pi}{12}\right)$

Pelo Teorema de Bolzano, conclui-se que existe um valor de $x \in \left]\frac{\pi}{12}; \frac{\pi}{5}\right[$ para o qual $a(x) = 5$

(9) $\lim f(x_n) = f(\lim x_n) = f(0^+) = \lim\limits_{x\to 0^+} \frac{\sin(-x)}{x} = \lim\limits_{x\to 0^+}\left(-\underbrace{\frac{\sin x}{x}}_{\text{L.N. } 1}\right) = -1$ (A)

$x_n = \frac{1}{n} \to 0^+$

(10) $g(x) = \sin(2x) - \cos(x)$

$g'(x) = 2\cos(2x) + \sin x$

$g'(a) = \frac{1}{2}$

$2\cos(2a) + \sin a = \frac{1}{2}$

$4\cos(2a) + 2\sin a = 1$

$4(\cos^2 a - \sin^2 a) + 2\sin a = 1$

$4\cos^2 a - 4\sin^2 a + 2\sin a - 1 = 0$

$4\cos^2 a - 4\sin^2 a + 2\sin a - 1 = 0$

$4(1 - \sin^2 a) - 4\sin^2 a + 2\sin a - 1 = 0$

$4 - 4\sin^2 a - 4\sin^2 a + 3 = 0$

$-8\sin^2 a + 2\sin a + 3 = 0$

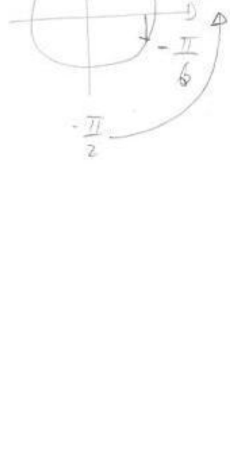
$8\sin^2 a - 2\sin a - 3 = 0$

$\sin a = \frac{2 \pm \sqrt{4 - 4(8)(-3)}}{2\times 8}$

$\sin a = \frac{2 \pm 10}{16}$

$\sin a = \frac{3}{4} \vee \sin a = -\frac{1}{2}$

imp. em $\left]-\frac{\pi}{2}, 0\right[$



$\therefore\ a = -\frac{\pi}{6}$

(11) f é cont. em $x=1$ sse $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

(11.1)

$\to \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1-\sqrt{x}+\sin(x-1)}{1-x} = \lim_{x\to 1^+} \frac{1-\sqrt{x}}{1-x} + \lim_{x\to 1^+} \frac{\sin(x-1)}{1-x}$

Calculemos os limites em separado.

• $\lim_{x\to 1^+} \frac{1-\sqrt{x}}{1-x} \overset{\frac{0}{0}}{=} \lim_{x\to 1^+} \frac{(1-\sqrt{x})(1+\sqrt{x})}{(1-x)(1+\sqrt{x})} = \lim_{x\to 1^+} \frac{1^2-(\sqrt{x})^2}{(1-x)(1+\sqrt{x})} =$

$= \lim_{x\to 1^+} \frac{\cancel{1-x}}{\cancel{(1-x)}(1+\sqrt{x})} = \lim_{x\to 1^+} \frac{1}{1+\sqrt{x}} = \frac{1}{1+1} = \frac{1}{2}$

• $\lim_{x\to 1^+} \frac{\sin((x-1))}{1-x} \overset{\frac{0}{0}}{=} \lim_{y\to 0} \frac{\sin y}{1-(y+1)} = \lim_{y\to 0} \frac{\sin y}{-y} = \lim_{y\to 0}\left(-\underbrace{\frac{\sin y}{y}}_{\substack{L.N. \\ 1}}\right) = -1$

$y = x-1 \to 0$

$x = y+1$

Assim $\lim_{x\to 1^+} f(x) = \frac{1}{2} - 1 = -\frac{1}{2}$

$\to \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(x e^{3+x} + 2x\right) = e^4 + 2$

$\to f(1) = e^4 + 2$

Logo f é descont. para $x=1$

(11.2)

$m = \lim_{x\to -\infty} \frac{xe^{3+x}+2x}{x} = \lim_{x\to -\infty} \left(\frac{xe^{3+x}}{x} + \frac{2x}{x}\right) = \lim_{x\to -\infty} \left(e^{3+x}+2\right) = e^{-\infty}+2$
$= 0+2$
$= 2$

$b = \lim_{x\to -\infty} \left(xe^{3+x} + \cancel{2x} - \cancel{2x}\right) = \lim_{x\to -\infty} \left(xe^{3+x}\right) \overset{\infty\times 0}{=}$

$= \lim_{\substack{y \\ -x\to +\infty}} \left(x\times e^3 \times e^x\right) = \lim_{y\to +\infty} \left(-y\times e^3 \times e^{-y}\right) = \lim_{y\to +\infty} \left(-e^3 \times \frac{y}{e^y}\right) =$

$y = -x \to +\infty$

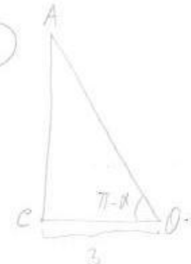
$x = -y$

$= \lim_{y\to +\infty} \left(-e^3 \times \frac{1}{\underbrace{\left(\frac{e^y}{y}\right)}_{\substack{L.N. \\ +\infty}}}\right) = -e^3 \times \frac{1}{+\infty} = -e^3 \times 0 = 0$

$\therefore y = 2x$ A.O.

(12)

(12.1)



$$tg(\pi-\alpha) = \frac{\overline{CA}}{3}$$

$$-tg\alpha = \frac{\overline{CA}}{3}$$

$$\boxed{\overline{CA} = -3tg\alpha}$$

$$\cos(\pi-\alpha) = \frac{3}{\overline{AO}}$$

$$-\cos\alpha = \frac{3}{\overline{AO}}$$

$$\boxed{\overline{AO} = -\frac{3}{\cos\alpha}}$$

Perímetro$_\triangle$ $= 2\overline{AO} + 2\overline{CA} = 2\left(-\frac{3}{\cos\alpha} - 3tg\alpha\right) = -6tg\alpha - \frac{6}{\cos\alpha}$ c.q.d.

(12.2) $P(x) = -6\,tg\,x - \frac{6}{\cos x}$

$$P'(x) = -6(tg\,x)' - \frac{6'\cos x - 6(\cos x)'}{\cos^2 x} =$$

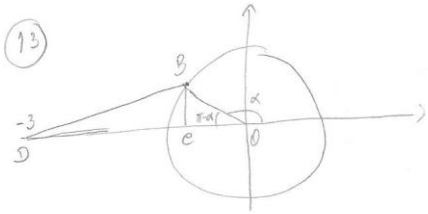
$$= -6 \times \frac{x'}{\cos^2 x} - \frac{0 + 6\sin x}{\cos^2 x} =$$

$$= -\frac{6}{\cos^2 x} - \frac{6\sin x}{\cos^2 x} =$$

$$= \frac{-6 - 6\sin x}{\cos^2 x}$$

$$m = P'\left(\frac{5\pi}{6}\right) = \frac{-6 - 6 \times \sin\left(\frac{5\pi}{6}\right)}{\cos^2\left(\frac{5\pi}{6}\right)} = \frac{-6 - 6 \times \frac{1}{2}}{\left(-\frac{\sqrt{3}}{2}\right)^2} = \frac{-9}{\frac{3}{4}} = -12$$

(13)



$$\sin(\pi-\alpha) = \frac{\overline{BC}}{1}$$

$$\boxed{\sin\alpha = \overline{BC}}$$

$$\cos(\pi-\alpha) = \frac{\overline{OC}}{1}$$

$$\boxed{-\cos\alpha = \overline{OC}}$$

$$\overline{DC} = \overline{DO} - \overline{OC} = $$
$$= 3 - (-\cos\alpha)$$
$$= 3 + \cos\alpha$$

$$A = \frac{\overline{DC} \times \overline{BC}}{2} =$$

$$= \frac{(3+\cos\alpha)\times \sin\alpha}{2}$$

$$= \frac{1}{2}(3+\cos\alpha)\sin\alpha \qquad (C)$$

(14) $f'(x) = x - \sin(2x)$

(14.1) $$\lim_{x\to\frac{\pi}{2}} \frac{f(x)-f(\frac{\pi}{2})}{2x-\pi} = \lim_{x\to\frac{\pi}{2}} \frac{f(x)-f(\frac{\pi}{2})}{2(x-\frac{\pi}{2})} = \frac{1}{2}\times \underbrace{\lim_{x\to\frac{\pi}{2}} \frac{f(x)-f(\frac{\pi}{2})}{x-\frac{\pi}{2}}}_{f'(\frac{\pi}{2})} =$$

$$= \frac{1}{2}\times f'\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2} - \sin\left(2\times\frac{\pi}{2}\right)\right) = \frac{1}{2}\left(\frac{\pi}{2} - \sin\pi\right) =$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

(14.2) $f''(x) = 1 - 2\cos(2x)$ em $\left]-\frac{\pi}{2}, \frac{\pi}{4}\right[$

Zeros: $1 - 2\cos(2x) = 0$
$\cos(2x) = \frac{1}{2}$

$2x = \frac{\pi}{3} + 2k\pi \vee 2x = -\frac{\pi}{3} + 2k\pi; k\in\mathbb{Z}$

$6x = \pi + 6k\pi \vee 6x = -\pi + 6k\pi; k\in\mathbb{Z}$

$x = \frac{\pi + 6k\pi}{6} \vee x = \frac{-\pi+6k\pi}{6}; k\in\mathbb{Z}$

$k=0 \leadsto x = \frac{\pi}{6} \vee x = -\frac{\pi}{6}$

$k=-1 \leadsto x = -\frac{5\pi}{6} \vee x = -\frac{7\pi}{6}$

$k=1 \leadsto x = \frac{7\pi}{6} \vee x = \frac{5\pi}{6}$

	$-\frac{\pi}{2}$		$-\frac{\pi}{6}$		$\frac{\pi}{6}$		$\frac{\pi}{4}$
f''	ND	+	0	-	0	+	ND
f	ND	$\cup$	pi	$\cap$	pi	$\cup$	ND

Conc. voltada p/ cima em $\left]-\frac{\pi}{2}; -\frac{\pi}{6}\right[$ e em $\left]\frac{\pi}{6}, \frac{\pi}{4}\right[$

" " " baixo em $\left]-\frac{\pi}{6}; \frac{\pi}{6}\right[$

f admite Pontos de Inflexão p/ $x = -\frac{\pi}{6}$ e p/ $x = \frac{\pi}{6}$

(15) Tem que verificar-se:

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right)$$

Ora, $\lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{x - \frac{\pi}{2}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{y \to 0} \frac{\cos\left(y + \frac{\pi}{2}\right)}{y} = \lim_{y \to 0} \frac{-\overbrace{\sin y}^{\text{L.N.} \to 1}}{y} = -1$

$y = x - \frac{\pi}{2} \to 0$

$x = y + \frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right) = K - 3$ Assim $K - 3 = -1 \Leftrightarrow K = 2$ (C)

(16) $\sin\alpha = \frac{\overline{QR}}{\overline{PR}} \Leftrightarrow$

$\Leftrightarrow \sin\alpha = \frac{\overline{QR}}{4}$

$\Leftrightarrow \boxed{\overline{QR} = 4\sin\alpha}$

$\cos\alpha = \frac{\overline{PQ}}{4}$

$\Leftrightarrow \boxed{\overline{PQ} = 4\cos\alpha}$

$A(\alpha) = \cancel{2} \times \frac{\overline{PQ} \times \overline{QR}}{\cancel{2}} =$

$= 4\sin\alpha \times 4\cos\alpha =$

$= 16\sin\alpha\cos\alpha$ c.q.d.

Agora,

$A(\theta) = 16\sin\theta\cos\theta$

$tg\,\theta = 2\sqrt{2}$

$1 + tg^2\theta = \frac{1}{\cos^2\theta}$

$1 + \left(2\sqrt{2}\right)^2 = \frac{1}{\cos^2\theta}$

$1 + 2^2 \times \left(\sqrt{2}\right)^2 = \frac{1}{\cos^2\theta}$

$1 + 8 = \frac{1}{\cos^2\theta}$

$9 = \frac{1}{\cos^2\theta}$

$\cos^2\theta = \frac{1}{9}$

$\cos\theta = \pm\frac{1}{3}$

Como $\theta \in \left]0, \frac{\pi}{2}\right[$, $\boxed{\cos\theta = \frac{1}{3}}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(\frac{1}{3}\right)^2 = 1$

$\sin^2\theta = 1 - \frac{1}{9}$

$\sin^2\theta = \frac{8}{9}$

$\sin\theta = \pm\frac{2\sqrt{2}}{3}$

Como $\theta \in \left]0, \frac{\pi}{2}\right[$, $\boxed{\sin\theta = \frac{2\sqrt{2}}{3}}$

Assim,

$A(\theta) = 16\sin\theta\cos\theta =$

$= 16 \times \frac{2\sqrt{2}}{3} \times \frac{1}{3} =$

$= \frac{32\sqrt{2}}{9}$