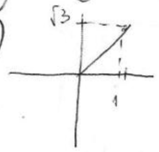
Resolução da ficha n.º 1 - Complexos - 12.º ANO

(1) $z = 1+\sqrt{3}i$ $\qquad \rho = \sqrt{1+3} = 2$

a)



$\left.\begin{array}{l}\cos\theta = \frac{1}{2}\\ \sin\theta = \frac{\sqrt{3}}{2}\end{array}\right\rangle \theta = \frac{\pi}{3}$

$z = 2e^{i\frac{\pi}{3}}$

$\sqrt[4]{2e^{i\frac{\pi}{3}}} = \sqrt[4]{2}\,e^{i\left(\frac{\frac{\pi}{3}+2k\pi}{4}\right)}$; $K = 0,1,2,3$

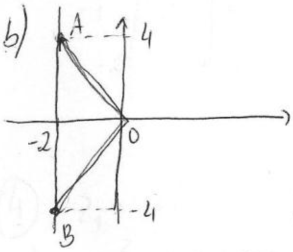
$K=0 \rightarrow z_1 = \sqrt[4]{2}\,e^{i\frac{\pi}{12}}$

$K=1 \rightarrow z_2 = \sqrt[4]{2}\,e^{i\frac{7\pi}{12}}$

$K=2 \rightarrow z_3 = \sqrt[4]{2}\,e^{i\frac{13\pi}{12}}$

$K=3 \rightarrow z_4 = \sqrt[3]{2}\,e^{i\frac{19\pi}{12}}$

b)



$Re(z) = -2 \Leftrightarrow x = -2$

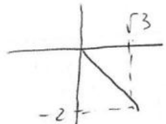
$A_\Delta = \frac{b\times h}{2} = \frac{b\times \cancel{2}}{\cancel{2}} = \boxed{b = 8}$

$z = -2+4i$

(2)

a) $\frac{(\sqrt{3}-2i)^2 + \left(2e^{i\frac{\pi}{9}}\right)^3}{\text{cis}\,\frac{3\pi}{2}} = \frac{(\sqrt{3}-2i)^2 + 8e^{i\frac{3\pi}{9}}}{-i} =$

$z = \sqrt{3}-2i$



$\rho = \sqrt{3+4} = \sqrt{7}$

$= \frac{(3-4\sqrt{3}i-4)+8e^{i\frac{\pi}{3}}}{-i} =$

$= \frac{(-1-4\sqrt{3}i)+\left(8\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right)}{-i} =$

$= \frac{(-1-4\sqrt{3}i)+8\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}{-i} = \frac{-1-\cancel{4\sqrt{3}i}+4+\cancel{4\sqrt{3}i}}{-i} =$

$= \frac{3}{-i} = \frac{3(-i)}{-1} = 3i$

b) $z_1 = e^{i\alpha}$ $\qquad z_2 = e^{i(\pi+\alpha)} = \cos(\pi+\alpha)+i\sin(\pi+\alpha) = -\cos\alpha - i\sin\alpha$

$z_2 = -e^{i\alpha}$

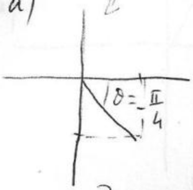
$\left.\begin{array}{l} z_1^3 = e^{i(3\alpha)} \\ z_2^3 = -e^{i(3\alpha)} \end{array}\right\rangle e^{i(3\alpha)} = -e^{i(3\alpha)} \Leftrightarrow 2e^{i(3\alpha)} = 0 \Leftrightarrow e^{i(3\alpha)}$

$\Leftrightarrow \left\{\begin{array}{l}\cos 3\alpha = 0 \rightarrow 3\alpha = \frac{\pi}{2}+K\pi \\ \sin 3\alpha = 0 \rightarrow 3\alpha = K\pi\end{array}\right\rangle \frac{\pi}{2}+K\pi = K\pi$

$\frac{\pi}{2} = 0$ imp.

③ $z_1 = 2-2i \qquad z_2 = \sqrt{2}\,e^{i\frac{5\pi}{4}} \qquad z_3 = -1+i$

a)



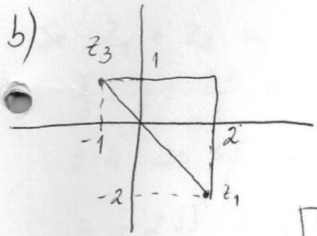
$\rho = \sqrt{4+4} = \sqrt{8}$

$z_1 = \sqrt{8}\,e^{i(-\frac{\pi}{4})} = 2\sqrt{2}\,e^{i(-\frac{\pi}{4})}$

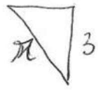
$\frac{z_1}{z_2} = \frac{2\sqrt{2}\,e^{i(-\frac{\pi}{4})}}{\sqrt{2}\,e^{i(\frac{5\pi}{4})}} = 2e^{i(-\frac{\pi}{4}-\frac{5\pi}{4})} = 2e^{i(-\frac{6\pi}{4})} =$

$= 2e^{i(-\frac{3\pi}{2})} = 2e^{i\frac{\pi}{2}} = 2i$

b)



raio = ?



$r^2 = 9+9$

$r^2 = 18$

$r = \sqrt{18} = 3\sqrt{2}$

$\boxed{|z - z_1| = 3\sqrt{2}}$

④ $z_1 = 1+i$

a) $z_1^2 + bz_1 + c = 0 \Leftrightarrow \cancel{1} + 2i - \cancel{1} + b + bi + c = 0 \Leftrightarrow$

$\Leftrightarrow b + c + (2+b)i = 0 \Leftrightarrow \begin{cases} b+c=0 \\ 2+b=0 \end{cases} \begin{cases} c=2 \\ b=-2 \end{cases}$

b) $z_2 = e^{i\alpha} \quad [0; 2\pi]$

$z_1 \times \bar{z}_2 = \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)\left(e^{i(-\alpha)}\right) =$

$= \sqrt{2}\,e^{i(\frac{\pi}{4}-\alpha)} =$

$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}-\alpha\right) + i\sin\left(\frac{\pi}{4}-\alpha\right)\right)$

$\begin{cases} \sin\left(\frac{\pi}{4}-\alpha\right) = 0 \\ \cos\left(\frac{\pi}{4}-\alpha\right) < 0 \end{cases} \begin{cases} \frac{\pi}{4} - \alpha = k\pi;\ k \in \mathbb{Z} \\ \frac{\pi}{2} < \frac{\pi}{4} - \alpha < \frac{3\pi}{2} \end{cases}$

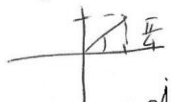
$\begin{cases} \alpha = \frac{\pi}{4} - k\pi \\ \frac{\pi}{4} < -\alpha < \frac{5\pi}{4} \end{cases} \begin{cases} \alpha = \frac{\pi}{4} + k\pi \\ -\frac{5\pi}{4} < \alpha < -\frac{\pi}{4} \end{cases}$

$k = 0 \rightarrow \alpha = \frac{\pi}{4}$ ✗

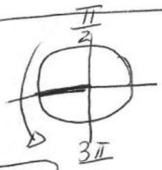
$k = -1 \rightarrow \alpha = -\frac{3\pi}{4}$

$z_1 = 1+i$

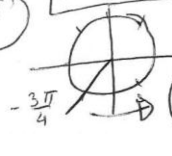
$\rho = \sqrt{1+1} = \sqrt{2}$



$z_1 = \sqrt{2}\,e^{i\frac{\pi}{4}}$



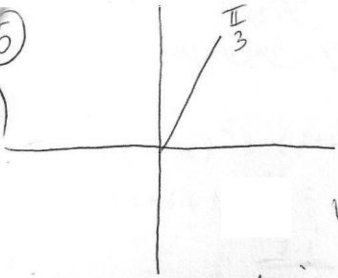
$\boxed{\cos < 0}$



$\frac{5\pi}{4} \in [0, 2\pi]$

5

a)



$z_1 = \rho_1 e^{i\frac{\pi}{3}}$

$z_2 = 4e^{i\theta}$

$w = \frac{-1+i}{i} = \frac{(-1+i)(-i)}{1} = \frac{1+i}{1} = 1+i$

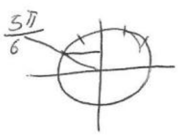
$w = 1+i \Rightarrow \rho = \sqrt{1+1} = \sqrt{2} \Rightarrow w \neq z_2$

$\theta = \frac{\pi}{4} \Rightarrow w \neq z_1$

b) $(z_1)^4 = \left(\rho_1 e^{i\left(\frac{\pi}{3}\right)}\right)^4 = \rho_1^4 e^{i\frac{4\pi}{3}}$

Se z_1 e z_2 são raízes quartas obtêm-se rodando $\frac{2\pi}{4} = \frac{\pi}{2}$

$\frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6} \in 2.^o$, logo $z_2 = 4e^{i\frac{5\pi}{6}} =$



$= 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) =$

$= -2\sqrt{3} + 2i$ //

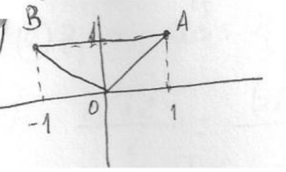
6 $z_1 = 1+i$ $\quad z_2 = \sqrt{2}e^{i\frac{3\pi}{4}}$

$\rho = \sqrt{1+1} = \sqrt{2}$

a) $z_1 = \sqrt{2}e^{i\frac{\pi}{4}}$ $\quad (z_1)^4 = (\sqrt{2})^4 e^{i(4\frac{\pi}{4})} = 4e^{i\pi} = -4$

$(z_2)^4 = (\sqrt{2})^2 e^{i(\frac{3\pi}{4}\times 4)} = 4e^{i(3\pi)} = -4$ c.q.d.

b)



$P = 2 + \sqrt{2} + \sqrt{2} = 2 + 2\sqrt{2}$ //

7) $w = 2+i$

a) $(w-2)^{11}(1+3i)^2 = (\cancel{2}+i-\cancel{2})^{11}(1+3i)^2 = i^{11}(1+3i)^2 = i^3(1+3i)^2 = -i(1+3i)^2 =$

$= -i(1+6i+9i^2) = -i(1+6i-9) = -i(-8+6i) = 8i - 6i^2 = 6+8i$

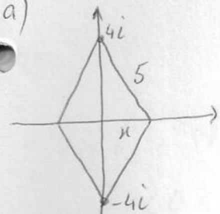
b) $\frac{1}{w} = \frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{4+1} = \frac{2}{5} - \frac{1}{5}i$

$\sqrt{2}e^{i\frac{3\pi}{4}} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -1+i$

R: O inverso de w não é $\sqrt{2}e^{i\frac{3\pi}{4}}$

8) $z_1 = 4i$

a)



$x^2 + 4^2 = 5^2$

$x^2 = 9$

$x = \pm 3$

c.s. $= \{-3; +3; 4i; -4i\}$

b) $\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^2 \cdot z = 2+4i \Leftrightarrow \left(2e^{i\frac{\pi}{2}}\right)\cdot z = 2+4i \Leftrightarrow$

$\Leftrightarrow (2i)\cdot z = 2+4i \Leftrightarrow z = \frac{2+4i}{2i} \Leftrightarrow z = \frac{(2+4i)(-2i)}{4} \Leftrightarrow$

$\Leftrightarrow z = \frac{-4i+8}{4} \Leftrightarrow z = 2-i$

9) $z_1 = 2e^{i\frac{\pi}{3}}$

a) $\frac{z_1^3+2}{i} = \frac{\left(2e^{i\frac{\pi}{3}}\right)^3+2}{i} = \frac{8e^{i\pi}+2}{i} = \frac{-8+2}{i} = \frac{-6}{i} =$

$= \frac{-6(-i)}{1} = 6i \rightarrow$ imaginário puro.

b) Circunf. de raio $= 2$

$\frac{\pi}{3} + \frac{2\pi}{5} = \frac{5\pi+6\pi}{15} = \frac{11\pi}{15}$

$|z| < 2 \wedge \frac{\pi}{3} < \arg(z) < \frac{11\pi}{15} \wedge z \neq 0$

(10) $z_1 = 7 + 24i$

a) Seja $z = 4 + yi$ o complexo que é uma das raízes quadradas de z_1.

$z^2 = z_1 \Leftrightarrow (4+yi)^2 = 7 + 24i \Leftrightarrow 16 + 8yi - y^2 = 7 + 24i \Leftrightarrow$

$\Leftrightarrow \begin{cases} 16 - y^2 = 7 \\ 8y = 24 \end{cases} \Leftrightarrow \begin{cases} 16 - 9 = 7 \\ y = 3 \end{cases} \Leftrightarrow y = 3$

A ordenada de P é 3

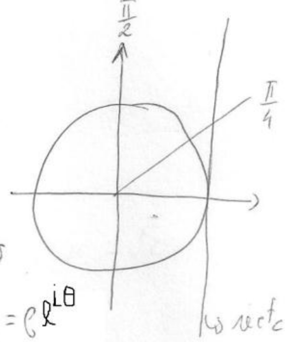
b) $z_2 = e^{i\alpha}$, $\alpha \in \left] \frac{3\pi}{4}, \pi \right[$

$z_1 = 7 + 24i$

$tg\,\theta = \frac{24}{7} > 1$

Como $z_1 \in 1.^\circ Q$ então

$\frac{\pi}{4} < \theta < \frac{\pi}{2}$ Seja $z_1 = \rho e^{i\theta}$



recta das tangentes

$z_1 \times z_2 = \rho e^{i\theta} \times e^{i\alpha} = \rho e^{i(\theta+\alpha)}$

$\frac{\pi}{4} < \theta < \frac{\pi}{2}$

$\frac{3\pi}{4} < \alpha < \pi$

$\frac{\pi}{4} + \frac{3\pi}{4} < \theta + \alpha < \frac{\pi}{2} + \pi$

$\pi < \theta + \alpha < \frac{3\pi}{2}$, logo $\pi < Arg(z_1 \times z_2) < \frac{3\pi}{2}$. Então a imagem geométrica de $z_1 \times z_2$ situa-se no $3.^\circ Q$.

(11) $z_1 = \rho e^{i\frac{\pi}{6}}$

a) $z_2 = \left(\rho e^{i\frac{\pi}{6}}\right)^4 = \rho^4 e^{i\frac{4\pi}{6}} = \rho^4 e^{i\frac{2\pi}{3}}$

$\frac{2\pi}{3} - \frac{\pi}{6} = \frac{4\pi - \pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$ c.q.d.

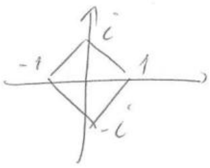
b) $P = 2\pi r = 4\pi \Leftrightarrow r = 2$. Logo $|z_1| = 2$

$z_1 = 2\, e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) =$

$= 2\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = \sqrt{3} + i$

(12)

a) $\frac{\overline{w}^2}{z} = \frac{(1-i)^2}{2e^{i\frac{3\pi}{2}}} = \frac{1-2i+i^2}{2e^{i\frac{3\pi}{2}}} = \frac{-2i}{-2i} = 1 = e^{i0}$



$\sqrt[4]{\frac{\overline{w}^2}{z}} = \sqrt[4]{e^{i0}} = \sqrt[4]{1}\, e^{i\frac{0+2k\pi}{4}}$, $k = 0, 1, 2, 3$

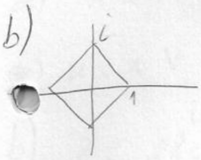
$k=0 \rightarrow z_0 = e^{i0} = 1$

$k=1 \rightarrow z_1 = e^{i\frac{\pi}{2}} = i$

$k=2 \rightarrow z_2 = e^{i\pi} = -1$

$k=3 \rightarrow z_3 = e^{i\frac{3\pi}{2}} = -i$

b)



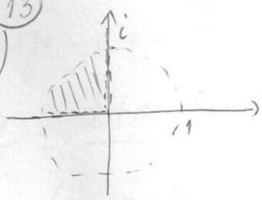
$r^2 = 1^2 + 1^2 \Leftrightarrow r = \sqrt{2}$

raio $= \frac{\sqrt{2}}{2}$

$|z| = \frac{\sqrt{2}}{2}$

(13)

a)



$|z| < 1 \wedge \frac{\pi}{2} < \arg(z) < \pi$

ou

$|z| < 1 \wedge Re(z) < 0 \wedge Im(z) > 0$

b) $1+\sqrt{3}\, i$

$tg\, \theta = \sqrt{3} \rightarrow \theta = \frac{\pi}{3}$

$\rho = \sqrt{1+3} = 2$

$\frac{1+\sqrt{3}i}{4e^{i\frac{\pi}{6}}} = \frac{2e^{i\frac{\pi}{3}}}{4e^{i\frac{\pi}{6}}} = \frac{1}{2}e^{i\left(\frac{\pi}{3}-\frac{\pi}{6}\right)} = \frac{1}{2}e^{i\frac{\pi}{6}}$ $\in 1^\circ Q$ e como $\frac{1}{2} < 1$ pertence ao conjunto A.

14

$P(x) = x^3 - 3x^2 + 6x - 4$

a) $P(x) = (x-1)(x^2 - 2x + 4)$

	1	-3	6	-4
1		1	-2	4
	1	-2	4	0

$P(x) = 0 \Leftrightarrow$

$\Leftrightarrow (x-1)(x^2 - 2x + 4) = 0 \Leftrightarrow$

$\Leftrightarrow x = 1 \vee x = \frac{2 \pm \sqrt{4-16}}{2}$

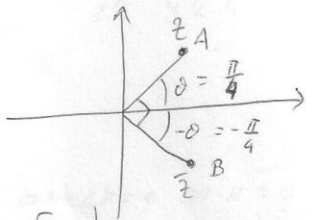
$\Leftrightarrow x = 1 \vee x = \frac{2 \pm \sqrt{-12}}{2} \Leftrightarrow x = 1 \vee x = \frac{2 \pm \sqrt{12}\, i}{2} \Leftrightarrow$

$\Leftrightarrow x = 1 \vee x = \frac{2 \pm 2\sqrt{3}\, i}{2} \Leftrightarrow x = 1 \vee x = \frac{1 \pm \sqrt{3}\, i}{1} \Leftrightarrow$

$\Leftrightarrow x = 1 \vee x = 1 + \sqrt{3}\, i \vee x = 1 - \sqrt{3}\, i$

b) $z = 2e^{i\theta}$

$\bar{z} = 2e^{i(-\theta)}$



Logo $z = 2e^{i\frac{\pi}{4}}$

$\frac{z}{i} = \frac{2\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)}{i} = \frac{2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)}{i} = \frac{\sqrt{2} + \sqrt{2}\, i}{i} =$

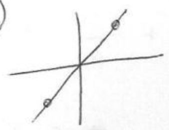
$= \frac{(\sqrt{2} + \sqrt{2}\, i)(-i)}{1} = \sqrt{2} - \sqrt{2}\, i$

(15) $w = a+bi$; $a>1$; $b>0$

$1-w = 1-(a+bi) = \underbrace{1-a}_{<0}\underbrace{-b}_{<0}i$, logo $1-w = z_3$ → Resposta (C)

(16) Resposta (B)

(17)

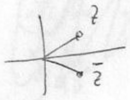


$w^4 = (\rho e^{i\frac{\pi}{4}})^4 = \rho^4 e^{i\pi} = -\rho^4$

$w^4 = (\rho e^{i\frac{5\pi}{4}})^4 = \rho^4 e^{i(5\pi)} = -\rho^4$

Eixo real (A)

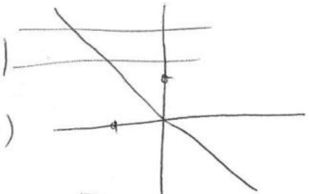
(18)



Resposta (B)

(19) $|z+1| = |z-i|$

$(-1,0)$ $(0,1)$



$2 \leq Im(z) \leq 4$

$2 \leq y \leq 4$

Resposta (B)

(20) $z+\bar{z} = 0 \Leftrightarrow x+yi+x-yi = 0 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0$

Resposta (A)

(21) $(a+bi)^2 = 3+4i \Leftrightarrow a^2-b^2+2abi = 3+4i$, logo

$2ab = 4 \Leftrightarrow ab = 2$. Então a e b têm o mesmo sinal.

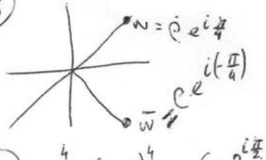
1.º Q ou 3.º Q

Resposta (A)

(22) Uma das raízes é $z = e^{i\frac{\pi}{2}}$

$z^7 = (e^{i\frac{\pi}{2}})^7 = e^{i\frac{7\pi}{2}} = e^{i\frac{3\pi}{2}} = -i$ Logo resposta (A)

(23)



$\frac{w}{\bar{w}} = \frac{\rho e^{i\frac{\pi}{4}}}{\rho e^{i(-\frac{\pi}{4})}} = e^{i\frac{\pi}{2}} = i = z_2$

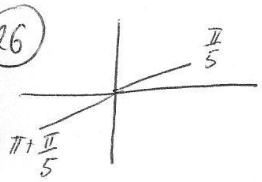
Resposta (B)

(24) $z^4 = (yi)^4 = (\rho e^{i\frac{\pi}{2}})^4 = \rho^4 e^{i(k\pi)} = \rho^4$ → Ponto A

Resposta (A)

(25) Resposta (B) → mediatriz

(26)



Resposta (C)

(27) $\frac{3\pi}{4} + \frac{2\pi}{6} = \frac{3\pi}{4} + \frac{\pi}{3} = \frac{13\pi}{12}$ Logo resposta (B)

→ "$\rho = \sqrt{2}$"

(28) $2\,i\,w = z_2$

i → 90°