Resolução da ficha n.º 3 - Complexos - 12.º ano

(1)
a) $z_1 = 3 + yi \quad z_2 = 4iz_1$

$z_1 = \rho e^{i\alpha}$

$-z_2 = -4iz_1 = 4e^{i\frac{3\pi}{2}} \rho e^{i\alpha} = 4\rho e^{i(\frac{3\pi}{2}+\alpha)}$

Logo $Arg(-z_2) = \frac{3\pi}{2} + \alpha$

b) $z_1 = 3 + yi$

$z_2 = 4iz_1 = 4i(3+yi) = 12i + 4yi^2 = -4y + 12i$

$Im(z_1) = Im(z_2) \Leftrightarrow$

$\Leftrightarrow y = 12$

$z_2 = -4 \times 12 + 12i = -48 + 12i$

(2)
a) $\overrightarrow{OA} = (\cos\alpha, \sin\alpha)$

$\overrightarrow{OB} = (\cos\alpha, -\sin\alpha)$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = (2\cos\alpha; 0)$

Assim, $w = 2\cos\alpha + 0i = 2\cos\alpha$

b) $\frac{z^3}{i} = \frac{(e^{i\alpha})^3}{e^{i\frac{\pi}{2}}} = \frac{e^{i(3\alpha)}}{e^{i\frac{\pi}{2}}} = e^{i(3\alpha - \frac{\pi}{2})}$

Para que seja real, terá que ser $\sin(3\alpha - \frac{\pi}{2}) = 0$,

ou seja, $3\alpha - \frac{\pi}{2} = k\pi \Leftrightarrow 3\alpha = \frac{\pi}{2} + k\pi \Leftrightarrow \alpha = \frac{\pi}{6} + \frac{k\pi}{3}$

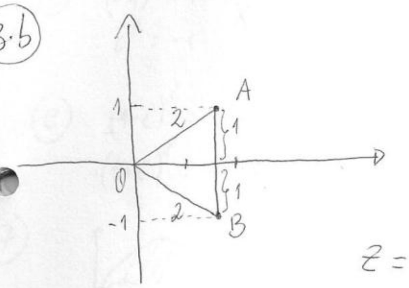
Sendo $\alpha \in]0; \frac{\pi}{2}[$ então $\alpha = \frac{\pi}{6}$

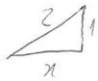
(3.a) $z_1 = (2-i)\left(2e^{i\frac{\pi}{2}}\right) \qquad z_2 = \frac{1}{5}e^{i(-\frac{\pi}{7})}$

$z_1 = (2-i)(2+i) = 2^2 - i^2 = 5$

$\frac{z_1}{z_2} = \frac{5}{\frac{1}{5}e^{i(-\frac{\pi}{7})}} = \frac{5e^{i0}}{\frac{1}{5}e^{i(-\frac{\pi}{7})}} = \frac{5}{\frac{1}{5}}e^{i(0+\frac{\pi}{7})} = 25e^{i(\frac{\pi}{7})}$

(3.b)





$x^2 + 1^2 = 2^2$
$x^2 + 1 = 4$
$x^2 = 4 - 1$
$x^2 = 3$
$x = \pm\sqrt{3}$

$z = \sqrt{3} + i$

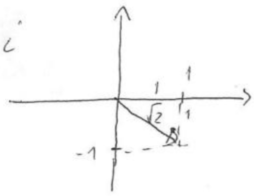
(4)

a) $\frac{4+2i\left(e^{i\frac{\pi}{6}}\right)^6}{3+i} = \frac{4+2i\left(e^{i\frac{6\pi}{6}}\right)}{3+i} = \frac{4+2i\left(e^{i\pi}\right)}{3+i} =$

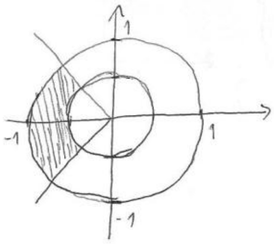
$= \frac{4+2i(-1)}{3+i} = \frac{4-2i}{3+i} = \frac{(4-2i)(3-i)}{3^2-i^2} = \frac{12-4i-6i+2i^2}{9+1} =$

$= \frac{12-10i-2}{10} = \frac{10-10i}{10} = 1-i$

$= \sqrt{2}e^{i(-\frac{\pi}{4})}$



b) $\frac{1}{2} \leq |z| \leq 1 \ \wedge \ \frac{3\pi}{4} \leq \arg(z) \leq \frac{5\pi}{4}$



Área $= \frac{1}{4}\left(A_{\text{círculo maior}} - A_{\text{círculo menor}}\right)$

$= \frac{1}{4}\left(\pi \times 1^2 - \pi \times \left(\frac{1}{2}\right)^2\right) =$

$= \frac{1}{4}\left(\pi - \frac{\pi}{4}\right) = \frac{1}{4}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{16}$

(5) $i^n = -i$

$i^{n+1} = i^n \times i = -i \times i = -i^2 = 1$ (A)

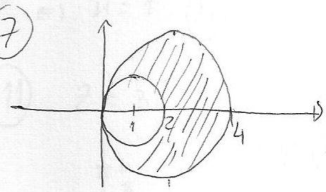
(6) (A) $1^2 = 1$
$i^2 = -1$ X

(B) $(-1)^2 = 1$
$i^2 = -1$ X

(C) $(1-i)^2 = 1^2 - 2i + i^2 = 1 - 2i - 1 = -2i$
$(1+i)^2 = 1^2 + 2i + i^2 = 1 + 2i - 1 = 2i$ X

(D) $(1-i)^2 = 1^2 - 2i + i^2 = 1 - 2i - 1 = -2i$ ✓
$(-1+i)^2 = (-1)^2 - 2i + i^2 = 1 - 2i - 1 = -2i$

(7)



$|z - (1+0i)| \geq 1 \wedge |z - (2+0i)| \leq 2$

$|z - 1| \geq 1 \wedge |z - 2| \leq 2$

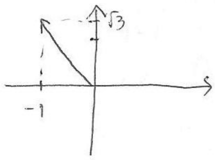
(8) $\sqrt{-i} = \sqrt{e^{i\frac{3\pi}{2}}} = e^{i\frac{\frac{3\pi}{2} + 2k\pi}{2}}$; $k = 0, 1$

Se $k = 0$ $z_1 = e^{i\frac{3\pi}{4}}$

Se $k = 1$ $z_2 = e^{i\frac{7\pi}{4}}$

(9) $z_1 = 1 - \sqrt{3}i$ $z_2 = 8e^{i0}$

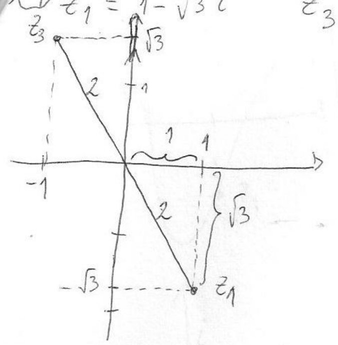
$-z_1 = -1 + \sqrt{3}i$ $\rho = \sqrt{1+3} = \sqrt{4} = 2$



$\begin{cases} \cos\theta = -\frac{1}{2} \\ \sin\theta = \frac{\sqrt{3}}{2} \end{cases}$ $\theta = \frac{2\pi}{3}$

$-z_1 = 2e^{i\frac{2\pi}{3}}$ $(-z_1)^3 = 2^3 e^{i(\frac{2\pi}{3} \times 3)} = 8e^{i(2\pi)} = 8e^{i0} = z_2$ c.q.d.

(9) $z_1 = 1 - \sqrt{3}i$ $\quad z_3 = (1-\sqrt{3}i)i^{46} = (1-\sqrt{3}i)i^2 = (1-\sqrt{3}i)(-1) = -1+\sqrt{3}i$



$\overline{AB} = 4$

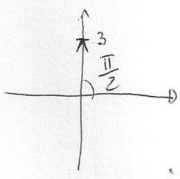
(10) $z + \bar{z} = 2 \Leftrightarrow$

$\Leftrightarrow x + yi + x - yi = 2 \Leftrightarrow$

$\Leftrightarrow 2x = 2 \Leftrightarrow$

$\Leftrightarrow x = 1$ (B)

(11) $z = 3i$

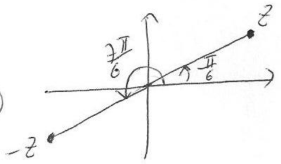


$\frac{1}{2}\pi$ (B)

(12) $z = e^{i\frac{\pi}{6}}$

$-z = e^{i\frac{7\pi}{6}}$

(D)



(14.a) $\frac{2z_1 - i^{18} - 3}{1-2i} = \frac{2-2i-i^2-3}{1-2i} =$

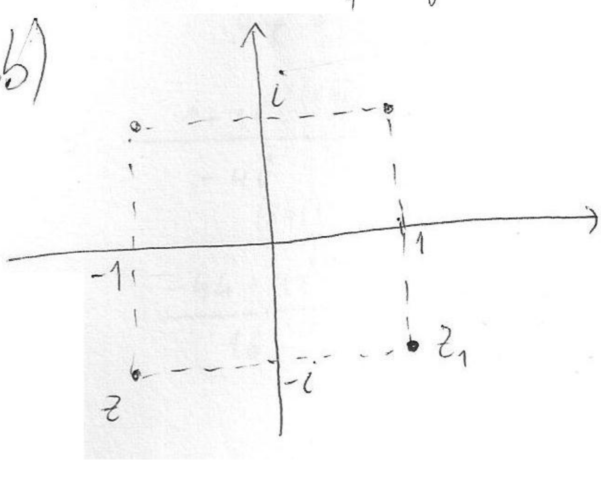
$= \frac{2-2i+1-3}{1-2i} = \frac{-2i(1+2i)}{1^2-4i^2} =$

$= \frac{-2i-4i^2}{1+4} = \frac{-2i+4}{5} =$

$= \frac{4}{5} - \frac{2}{5}i$

(13) $Re(z) < 3 \wedge -\frac{\pi}{4} < arg(z) < 0$ (A)

14.b)



$z = -1 - i = \sqrt{2}e^{i\frac{5\pi}{4}}$

(15) $z_1 = (K-i)(3-2i) = 3K - 2Ki - 3i + 2i^2 = (3K-2) + (-2K-3)i$ ($i^2 = -1$)

z_1 é imaginário puro se $3K-2=0$ ou seja $K = \frac{2}{3}$ (C)

(16) (A)

(17) $\dfrac{\left(e^{i\frac{\pi}{7}}\right)^7 + (2+i)^3}{4e^{i\frac{3\pi}{2}}} = \dfrac{-1+2+11i}{-4i} =$

$= \dfrac{1+11i}{-4i} \cdot \dfrac{(\times 4i)}{(\times 4i)} = \dfrac{4i + 44i^2}{-16i^2} =$ ($i^2 = -1$)

$= \dfrac{-44+4i}{16} = -\dfrac{11}{4} + \dfrac{1}{4}i$

C.A.

$(2+i)^3 = (2+i)^2(2+i) =$

$= (4+4i+i^2)(2+i) =$

$= (3+4i)(2+i) =$

$= 6+3i+8i+4i^2 =$

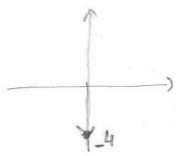
$= 2+11i$

$\left(e^{i\left(\frac{\pi}{7}\right)}\right)^7 = e^{i\left(\frac{7\pi}{7}\right)} =$

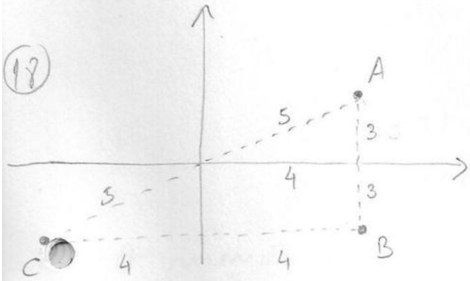
$= e^{i\pi} = \cos\pi + i\sin\pi =$

$= -1$

$4e^{i\left(\frac{3\pi}{2}\right)} = -4i$



(18)



$x^2 + 4^2 = 5^2$

$x = \pm 3$

$A_\Delta = \dfrac{8 \times 6}{2} = 24$

(19)

$z = \rho e^{i\frac{\pi}{3}}$

$\bar{z} = \rho e^{i(-\frac{\pi}{3})}$

$2i = 2e^{i\frac{\pi}{2}}$

$\dfrac{2i}{\bar{z}} = \dfrac{2e^{i\frac{\pi}{2}}}{\rho e^{i(-\frac{\pi}{3})}} = \dfrac{2}{\rho} e^{i\left(\frac{\pi}{2} + \frac{\pi}{3}\right)} = \dfrac{2}{\rho} e^{i\frac{5\pi}{6}}$

(C)

(20) $z_1 = bi$ (C)

$(z_1)^2 = (bi)^2 = b^2 i^2 = -b^2$

$(z_1)^3 = (bi)^3 = b^3 i^3 = -b^3 i$

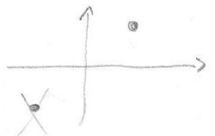
(21) $z_1 = \frac{i}{1-i} - i^{18}$ $\qquad z_2 = e^{i\frac{5\pi}{6}}$

a) $z_1 = \frac{i(1+i)}{(1-i)(1+i)} - i^2 = \frac{i+i^2}{1-i^2} + 1 = \frac{-1+i}{1+1} + 1 = \frac{-1+i}{2_{(1)}} + \frac{1}{(2)} = \frac{-1+i+2}{2}$

$= \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$

$\rho = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}_{(\times\sqrt{2})}^{(\times\sqrt{2})} = \frac{\sqrt{2}}{2}$

$tg\,\theta = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$



$\theta = \frac{\pi}{4}$

$z_1 = \frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}}$

C.A.:

$-i = e^{i(-\frac{\pi}{2})}$

$-1 = e^{i\pi}$

$(-i z_2)^n = -1$

$\left[e^{i(-\frac{\pi}{2})} \times e^{i\frac{5\pi}{6}}\right]^n = -1$

$\left[e^{i(-\frac{\pi}{2}+\frac{5\pi}{6})}\right]^n = -1$

$\left(e^{i\frac{\pi}{3}}\right)^n = -1$

$e^{i\left(\frac{n\pi}{3}\right)} = e^{i\pi}$

$\Leftrightarrow \frac{n\pi}{3} = \pi + 2k\pi \Rightarrow n\pi = 3\pi + 6k\pi \Leftrightarrow$

$\Leftrightarrow n = 3 + 6k$

O menor n é 3 (para $k = 0$)

(22) $n = 5$

A: 0

B: $0 + \frac{2\pi}{5} = \frac{2\pi}{5}$

C: $\frac{2\pi}{5} + \frac{2\pi}{5} = \frac{4\pi}{5}$

D: $\frac{4\pi}{5} + \frac{2\pi}{5} = \frac{6\pi}{5}$ $\quad$ (B)

(23) $w = \rho e^{i\frac{3\pi}{2}}$

$w^6 = \rho^6 e^{i\left(\frac{18\pi}{2}\right)} = \rho^6 e^{i(9\pi)}$ eixo real $\quad$ (A)

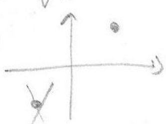
(24) $z_1 = \sqrt{2}\,e^{i\frac{\pi}{4}}$ $\qquad z_2 = 3$

a) $W = \dfrac{z_1^4 + 4i}{i} = \dfrac{-4+4i}{i}\cdot\dfrac{(-i)}{(-i)} = \dfrac{4i - 4i^2}{-i^2} = \dfrac{4+4i}{1} = 4+4i$

c.A.: $z_1^4 = \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^4 = (\sqrt{2})^4 e^{i\left(\frac{4\pi}{4}\right)} = 4e^{i\pi} = -4$

$\rho = \sqrt{4^2+4^2} = \sqrt{32}$

$\operatorname{tg}\theta = \frac{4}{4} = 1$



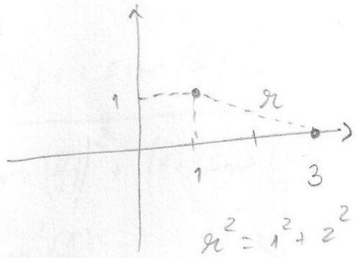
$\theta = \frac{\pi}{4}$

$W = \sqrt{32}\,e^{i\frac{\pi}{4}}$

b) $z_1 = \sqrt{2}\,e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) =$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 1+i$

$|z-3| = \sqrt{5}$



$r^2 = 1^2 + 2^2$

$r = \sqrt{5}$

(25) $z = 3e^{i\left(\frac{\pi}{8}-\theta\right)}$

z é imaginário puro se $\cos\left(\frac{\pi}{8}-\theta\right) = 0$, ou seja,

se $\frac{\pi}{8} - \theta = \frac{\pi}{2} + k\pi$

$\theta = \frac{\pi}{8} - \frac{\pi}{2} - k\pi$

$\theta = -\frac{3\pi}{8} - k\pi$

se $k=-1$, $\theta = -\frac{3\pi}{8} + \pi = \frac{5\pi}{8}$ (D)

(26) (B)

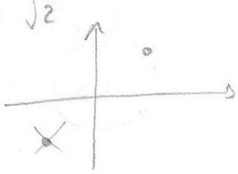
27) $z_1 = e^{i\frac{\pi}{7}}$ $\qquad z_2 = 2+i$

a) $$W = \frac{3 - i\times(z_1)^7}{\overline{z_2}} = \frac{3 - i\times\left(e^{i\frac{\pi}{7}}\right)^7}{2-i} = \frac{3 - i\times e^{i\pi}}{2-i} =$$

$$= \frac{3 - i\times(-1)}{2-i} = \frac{3+i}{2-i} = \frac{(3+i)(2+i)}{(2-i)(2+i)} = \frac{6+3i+2i+i^2}{4-i^2} =$$

$$= \frac{5+5i}{5} = 1+i$$

$$\rho = \sqrt{1^2+1^2} = \sqrt{2}$$

$tg\,\theta = \frac{1}{1} = 1$  $\theta = \frac{\pi}{4}$

$$W = \sqrt{2}\,e^{i\frac{\pi}{4}}$$

b) $$|z_1+z_2|^2 = \left|e^{i\frac{\pi}{7}} + 2 + i\right|^2 = \left|\cos\left(\frac{\pi}{7}\right) + i\sin\left(\frac{\pi}{7}\right) + 2 + i\right|^2 =$$

$$= \left|\left(2+\cos\left(\frac{\pi}{7}\right)\right) + \left(1+\sin\left(\frac{\pi}{7}\right)\right)i\right|^2 = \left(\sqrt{\left(2+\cos\left(\frac{\pi}{7}\right)\right)^2 + \left(1+\sin\left(\frac{\pi}{7}\right)\right)^2}\right)^2 =$$

$$= 4 + 4\cos\left(\frac{\pi}{7}\right) + \cos^2\left(\frac{\pi}{7}\right) + 1 + 2\sin\left(\frac{\pi}{7}\right) + \sin^2\left(\frac{\pi}{7}\right) =$$

$$= 4 + 1 + \underbrace{\cos^2\left(\frac{\pi}{7}\right) + \sin^2\left(\frac{\pi}{7}\right)}_{=1} + 4\cos\left(\frac{\pi}{7}\right) + 2\sin\left(\frac{\pi}{7}\right) =$$

$$= 6 + 4\cos\left(\frac{\pi}{7}\right) + 2\sin\left(\frac{\pi}{7}\right)$$ c.q.d.