Ficha 4 - Complexos - Resolução

(1) $i^{4n}+i^{4n+1}+i^{4n+2} = \underbrace{(i^4)^n}_{1} + \underbrace{(i^4)^n \times i}_{1\times i} + \underbrace{(i^4)^n\times i^2}_{1\times(-1)} = \cancel{1}+i-\cancel{1} = i \to z_2$ (B)

(2) $\measuredangle AOB = \frac{2\pi}{5}$ raio $=\sqrt[5]{32}=2$

$A_{\triangle} = \frac{\alpha r^2}{2}$. Assim, $A = \frac{\frac{2\pi}{5}\times 2^2}{2} = \frac{\frac{8\pi}{5}}{\frac{2}{1}} = \frac{8\pi}{10} = \frac{4\pi}{5}$ (B)

(3)

(3.1)

	1	-1	16	-16
1		1	0	16
	1	0	16	0 = Resto

$z^2+16=0$

$z^2=-16$

$z=\pm\sqrt{-16}$

$z=\pm\sqrt{16}\,i$

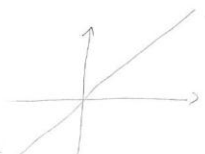
$z=\pm 4i$

R: $z=4e^{i\frac{\pi}{2}} \vee z = 4e^{i(-\frac{\pi}{2})}$

(3.2) $z_2 = 5i = 5e^{i\frac{\pi}{2}}$

$z_3 = e^{i(\frac{n\pi}{40})}$

$z_2\times z_3 = 5e^{i(\frac{\pi}{2}+\frac{n\pi}{40})} = 5e^{i\left(\frac{20\pi+n\pi}{40}\right)}$



$\frac{5\pi}{4}$

$\frac{20\pi+n\pi}{40_{(1)}} = \frac{5\pi}{4_{(10)}} + 2k\pi_{(40)}\ ; k\in\mathbb{Z}$

$20\pi+n\pi = 50\pi+80k\pi\ ; k\in\mathbb{Z}$

$n\pi = 30\pi+80k\pi\ ; k\in\mathbb{Z}$

$n = 30+80k\ ; k\in\mathbb{Z}$

$k=0 \Rightarrow n=30$

R: $n=30$

(D)

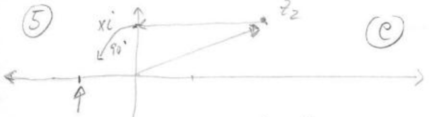
(4) $z=-\sqrt3+i$

$\rho = \sqrt{(-\sqrt3)^2+1^2}=2$

$tg\,\theta = \frac{1}{-\sqrt3} = -\frac{\sqrt3}{3} \to \theta=\frac{5\pi}{6}$

$|z|\le 2 \wedge \frac{5\pi}{6}\le \arg(z)\le\pi$ (B)

(5)



(C)

(6.1) $w = \frac{z_1\times i^{4n+3} - b}{\sqrt2\, e^{i(\frac{5\pi}{4})}} = \frac{(1+2i)\overbrace{(i^4)^n}^{1}\times \overbrace{i^3}^{-i} - b}{\sqrt2\left(\cos(\frac{5\pi}{4})+i\sin(\frac{5\pi}{4})\right)} = \frac{-i(1+2i)-b}{\sqrt2\left(-\frac{\sqrt2}{2}-\frac{\sqrt2}{2}i\right)} =$

$= \frac{-i-2i^2-b}{-1-i} = \frac{-i+2-b}{-1-i} = \frac{(-i+2-b)(-1+i)}{(-1)^2-i^2} = \frac{i-2+b-i^2+2i-bi}{1+1} = \frac{b-1+3i-bi}{2}$

$= \frac{b-1}{2}+\frac{3-b}{2}i$ Para que w seja real: $\frac{3-b}{2}=0 \Leftrightarrow 3-b=0 \Leftrightarrow b=3$ //

6.2 $|z| = 1 \Leftrightarrow \sqrt{a^2+b^2} = 1 \Leftrightarrow a^2+b^2 = 1$ $z = a+bi$

$|1+z|^2 + |1-z|^2 = |1+a+bi|^2 + |1-a-bi|^2 = \left(\sqrt{(1+a)^2+b^2}\right)^2 + \left(\sqrt{(1-a)^2+(-b)^2}\right)^2 =$

$= 1 + \cancel{2a} + \underbrace{a^2+b^2}_{1} + 1 - \cancel{2a} + \underbrace{a^2+b^2}_{1} = 1+1+1+1 = 4$

7 $\frac{w\,(\times 3i)}{3i\,(\times 3i)} = \frac{3wi}{9i^2} = \frac{wi}{3(-1)} = -\frac{w}{3}i = \frac{w}{3}\times(-i)$ $-90°$

O módulo de w reduz para a terça parte e o afixo roda de $90°$ no sentido negativo (Z_1) (A)

8 C

9.1 $z^3 + z_1 = z_2 \Leftrightarrow z^3 = z_2 - z_1 \Leftrightarrow$

$z^3 = 6+11i + 2 - 11i \Leftrightarrow$

$\Leftrightarrow z^3 = 8$

$\Leftrightarrow z^3 = 8e^{i0}$

$\Leftrightarrow z = \sqrt[3]{8}\, e^{i\left(\frac{0+2k\pi}{3}\right)}, k = 0,1,2$

$k=0 \rightarrow z = 2e^{i0}$

$k=1 \rightarrow z = 2e^{i\frac{2\pi}{3}}$

$k=2 \rightarrow z = 2e^{i\frac{4\pi}{3}}$

$z_1 = (-2+i)^3 = (-2+i)(-2+i)^2 = (-2+i)(4-4i+i^2) = (-2+i)(3-4i) =$

$= -6+8i+3i-4i^2 = -2+11i$

$z_2 = \frac{1+28i}{2+i} = \frac{(1+28i)(2-i)}{(2+i)(2-i)} =$

$= \frac{2-i+56i-28i^2}{4-i^2} = \frac{30+55i}{5} = 6+11i$

9.2 $w = \rho e^{i\theta}$

$w^n = \left(\frac{1}{w}\right)^n \Leftrightarrow \left(\rho e^{i\theta}\right)^n = \left(\frac{1}{\rho e^{i\theta}}\right)^n \Leftrightarrow \rho^n e^{i(n\theta)} = \left(\frac{e^{i0}}{\rho e^{i\theta}}\right)^n$

$\Leftrightarrow \rho^n e^{i(n\theta)} = \left(\frac{1}{\rho} e^{i(0-\theta)}\right)^n \Leftrightarrow \rho^n e^{i(n\theta)} = \frac{1}{\rho^n} e^{i(-n\theta)} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \rho^n = \frac{1}{\rho^n} \\ n\theta = -n\theta + 2k\pi;\ k\in\mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho^{2n} = 1 \\ \cancel{2}n\theta = \cancel{2}k\pi;\ k\in\mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho = 1 \\ \theta = \frac{k\pi}{n};\ k\in\mathbb{Z} \end{cases}$

$z = w^n = \left(e^{i\left(\frac{k\pi}{n}\right)}\right)^n = e^{i\left(\frac{\cancel{n}k\pi}{\cancel{n}}\right)} = e^{i(k\pi)};\ k\in\mathbb{Z}$

Se k par, $z = 1$

se k ímpar, $z = -1$

Ou sabe-se que $z = w^n$ e $z = \left(\frac{1}{w}\right)^n$

Assim, $w^n = \left(\frac{1}{w}\right)^n \Leftrightarrow w^n = \frac{1}{w^n} \Leftrightarrow (w^n)^2 = 1 \Leftrightarrow z^2 = 1 \Leftrightarrow z = \pm\sqrt{1} \Leftrightarrow z = 1 \vee z = -1$

(10) $z_1 \times \overline{z_2} = (2+i)\times(3+Ki) = 6+2Ki+3i+K i^2 = (6-K)+(3+2K)i$ (D)

Para que $z_1 \times \overline{z_2}$ seja imaginário puro, $6-K=0 \Leftrightarrow K=6$

(11) $n=9$ $\quad \frac{3\pi}{2} - 4\times\frac{2\pi}{9} = \frac{3\pi}{2} - \frac{8\pi}{9} = \frac{11\pi}{18}$ (B)

$\rho = 3$

$\theta = \frac{2\pi}{9}$

(12)

(12.1) $$\frac{\sqrt{3}\times i^{4n-6} + 2\,\text{cis}\left(-\frac{\pi}{6}\right)}{2e^{i\frac{\pi}{5}}} = \frac{\sqrt{3}\times (i^4)^n \times i^{-6} + 2\left(\cos\left(-\frac{\pi}{6}\right) + i\,\text{sim}\left(-\frac{\pi}{6}\right)\right)}{2e^{i\frac{\pi}{5}}} =$$

$$= \frac{\sqrt{3}\times 1^n \times (i^6)^{-1} + 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)}{2e^{i\frac{\pi}{5}}} = \frac{\sqrt{3}\times 1 \times (i^2)^{-1} + \sqrt{3} - i}{2e^{i\frac{\pi}{5}}} = \frac{\sqrt{3}\times(-1)^{-1} + \sqrt{3} - i}{2e^{i\frac{\pi}{5}}} =$$

$$= \frac{-\sqrt{3}+\sqrt{3}-i}{2e^{i\frac{\pi}{5}}} = \frac{e^{i\frac{3\pi}{2}}}{2e^{i\frac{\pi}{5}}} = \frac{1}{2}e^{i\left(\frac{3\pi}{2}-\frac{\pi}{5}\right)} = \frac{1}{2}e^{i\left(\frac{13\pi}{10}\right)}$$

(12.2) $$z_1 + z_2 = e^{i\alpha} + e^{i\left(\alpha+\frac{\pi}{2}\right)} = \cos\alpha + i\,\text{sim}\,\alpha + \underbrace{\cos\left(\alpha+\frac{\pi}{2}\right)}_{-\text{sim}\,\alpha} + i\,\underbrace{\text{sim}\left(\alpha+\frac{\pi}{2}\right)}_{\cos\alpha} =$$

$$= \underbrace{\left(\cos\alpha - \text{sim}\,\alpha\right)}_{<0} + i\,\underbrace{\left(\overset{+}{\text{sim}\,\alpha} + \overset{+}{\cos\alpha}\right)}_{>0} \in 2^{\circ}Q \qquad \alpha \in \left]\frac{\pi}{4}, \frac{\pi}{2}\right[$$

porque $\cos\alpha < \text{sim}\,\alpha$ $\quad$ porque $\text{sim}\,\alpha > 0$ e $\cos\alpha > 0$

(13) $$i^{8n} \times i^{8n-1} + i^{8n-2} = (i^8)^n \times (i^8)^n \times i^{-1} + (i^8)^n \times i^{-2} =$$ (C)

$$= 1^n \times 1^n \times \frac{1}{i} + 1^n \times \frac{1}{i^2} = \frac{1}{i} + \frac{1}{i^2} = \frac{i}{i^2} + \frac{1}{-1} = \frac{i}{-1} - 1 = -i-1$$

(14) $z = -8+6i < \begin{matrix}\rho = \sqrt{64+36} = 10 \\ \arg z = \alpha\end{matrix} > z = 10e^{i\alpha}$

$$w = \frac{-i\times z^2}{\overline{z}} = \frac{e^{i\left(-\frac{\pi}{2}\right)}\times\left(10e^{i\alpha}\right)^2}{10e^{i(-\alpha)}} = \frac{e^{i\left(-\frac{\pi}{2}\right)}\times 100e^{i(2\alpha)}}{10e^{i(-\alpha)}} =$$

$$= \frac{100e^{i\left(2\alpha-\frac{\pi}{2}\right)}}{10e^{i(-\alpha)}} = 10e^{i\left(2\alpha-\frac{\pi}{2}+\alpha\right)} = 10e^{\left(3\alpha-\frac{\pi}{2}\right)}$$ (A)

(15.1) $z_1 = \sqrt{2} + 2e^{i\frac{3\pi}{4}} = \sqrt{2} + 2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2} + 2\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) =$

$= \cancel{\sqrt{2}} - \cancel{\sqrt{2}} + \sqrt{2}\, i = \sqrt{2}\, i = \sqrt{2}e^{i\frac{\pi}{2}}$

$z_2 = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

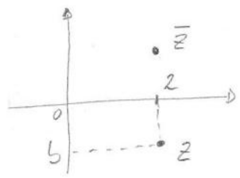
$\left.\begin{array}{l} \rho = \sqrt{1+1} = \sqrt{2} \\ tg\,\theta = \frac{1}{1} = 1 \end{array}\right\}$ $\quad w = \left(\frac{z_1}{z_2}\right)^4 = \left(\frac{\sqrt{2}e^{i\frac{\pi}{2}}}{\sqrt{2}e^{i\frac{\pi}{4}}}\right)^4 = \left(e^{i\left(\frac{\pi}{4}\right)}\right)^4 = e^{i\pi} = -1$

(15.2) $z_3 + \overline{z_2} = \cos\alpha + i\sin\alpha + 1 - i = (\cos\alpha + 1) + (-1 + \sin\alpha)i$

$-1 + \sin\alpha = 0 \Leftrightarrow \sin\alpha = 1 \Leftrightarrow \alpha = \frac{\pi}{2} + 2k\pi \,;\, k \in \mathbb{Z}$

$k = -1 \to \alpha = -\frac{3\pi}{2}$

(16)



$\arg = \alpha$

$\rho > 2$ $\quad$ logo $\quad 3e^{i\alpha}$ $\quad$ (C)

(17) $\frac{3}{2} \leq |z - 3 + i| \leq 3 \;\wedge\; \frac{\pi}{3} \leq \arg(z - 3 + i) \leq \frac{2\pi}{3}$ $\quad$ (A)

$\frac{3}{2} \leq |z - (3 - i)| \leq 3 \;\wedge\; \frac{\pi}{3} \leq \arg(z - (3 - i)) \leq \frac{2\pi}{3}$

(18.1) $z_1 = \frac{1 + \sqrt{3}\, i}{2} + i^{22} = \frac{1 + \sqrt{3}\, i}{2} - 1 = \frac{-1 + \sqrt{3}\, i}{2}$

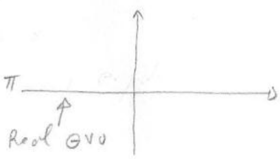
$i^2 = -1$

$i z_1 = i\left(\frac{-1 + \sqrt{3}\, i}{2}\right) = \frac{\sqrt{3}}{2}i^2 - \frac{1}{2}i = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$z_2 = \frac{-2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)}{\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)} = \frac{\sqrt{3} - i}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}i\right)^2} = \frac{\sqrt{3} - i}{\frac{3}{4} + \frac{1}{4}} = \frac{\sqrt{3} - i}{1} = \sqrt{3} - i$

$z_2 = \sqrt{3} - i \left\langle \begin{array}{l} \rho = \sqrt{3+1} = 2 \\ tg\,\theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \end{array} \right\rangle z_2 = 2e^{i\left(-\frac{\pi}{6}\right)}$

$(z_2)^n = \left(2e^{i\left(-\frac{\pi}{6}\right)}\right)^n = 2^n e^{i\left(-\frac{n\pi}{6}\right)}$



Assim,

$-\frac{n\pi}{6} = \pi + 2k\pi \,;\, k \in \mathbb{Z} \Leftrightarrow -n\pi = 6\pi + 12k\pi \,;\, k \in \mathbb{Z}$

$\Leftrightarrow n = -6 - 12k \,;\, k \in \mathbb{Z}$

se $k = -1 \to n = 6$ é o menor natural

(18.2) $\frac{\cos(\pi - \alpha) + i\cos\left(\frac{\pi}{2} - \alpha\right)}{\cos\alpha + i\sin\alpha} = \frac{\cos(\pi - \alpha) + i\sin\alpha}{e^{i\alpha}} = \frac{\cos(\pi - \alpha) + i\sin(\pi - \alpha)}{e^{i\alpha}} =$

$\frac{e^{i(\pi - \alpha)}}{e^{i\alpha}} = e^{i(\pi - \alpha - \alpha)} = e^{i(\pi - 2\alpha)}$ c.q.d.

(19) $z_1 = -2\sqrt{2} + 2\sqrt{2}\,i \longrightarrow z_1 = 4e^{i\frac{3\pi}{4}}$

$\rho = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$

$\left.\begin{array}{l} tg\,\theta = \frac{2\sqrt{2}}{-2\sqrt{2}} = -1 \\ \theta \in 2.^{\circ} Q \end{array}\right\} \theta = \frac{3\pi}{4}$

Então $w = 4e^{i\left(\frac{3\pi}{4} + 2\times\frac{2\pi}{6}\right)}$

$w = 4e^{i\left(\frac{17\pi}{12}\right)}$ (D)

(20) Seja $w = -1 + \sqrt{3}\,i$

(20.1) $\rho = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\left.\begin{array}{l} tg\,\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3} \\ \theta \in 2.^{\circ} Q \end{array}\right\} \theta = \frac{2\pi}{3}$

$z_1 = \frac{\left(2e^{i\left(\frac{2\pi}{3}\right)}\right)^3}{\sqrt{2}\,e^{i\left(-\frac{\pi}{4}\right)}} = \frac{8e^{i(2\pi)}}{\sqrt{2}\,e^{i\left(-\frac{\pi}{4}\right)}} = \frac{8}{\sqrt{2}}e^{i\left(2\pi + \frac{\pi}{4}\right)} =$

$= 4\sqrt{2}\,e^{i\left(\frac{9\pi}{4}\right)}$

$z_1 \times (z_2)^2 = 4\sqrt{2}\,e^{i\left(\frac{9\pi}{4}\right)} \times \left(e^{i\alpha}\right)^2 = 4\sqrt{2}\,e^{i\frac{9\pi}{4}} \times e^{i(2\alpha)} = 4\sqrt{2}\,e^{i\left(\frac{9\pi}{4} + 2\alpha\right)}$

Para que $z_1 \times (z_2)^2$ seja imaginário puro,

$\frac{9\pi}{4} + 2\alpha = \frac{\pi}{2} + K\pi \,;\, K \in \mathbb{Z} \Leftrightarrow 2\alpha = \frac{\pi}{2} - \frac{9\pi}{4} + K\pi \,;\, K \in \mathbb{Z} \Leftrightarrow 2\alpha = -\frac{7\pi}{4} + K\pi \,;\, K \in \mathbb{Z}$

$\Leftrightarrow \alpha = -\frac{7\pi}{8} + \frac{K\pi}{2} \,;\, K \in \mathbb{Z}$

$K = 0 \longrightarrow \alpha = -\frac{7\pi}{8}$ x

$K = 1 \longrightarrow \alpha = -\frac{3\pi}{8}$ x

$K = 2 \longrightarrow \alpha = \frac{\pi}{8}$

$K = 3 \longrightarrow \alpha = \frac{5\pi}{8}$

$K = 4 \longrightarrow \alpha = \frac{9\pi}{8}$ x

$S = \left\{\frac{\pi}{8} \,;\, \frac{5\pi}{8}\right\}$

(20.2) Sabe-se que:

$z = x + yi$

$|1 + z|^2 + |1 - z|^2 < 10$

$|1 + x + yi|^2 + |1 - x - yi|^2 < 10$

$\Leftrightarrow |(1 + x) + yi|^2 + |(1 - x) + (-y)i|^2 < 10$

$\Leftrightarrow \left(\sqrt{(1 + x)^2 + y^2}\right)^2 + \left(\sqrt{(1 - x)^2 + (-y)^2}\right)^2 < 10$

$\Leftrightarrow 1 + \cancel{2x} + x^2 + y^2 + 1 - \cancel{2x} + x^2 + y^2 < 10$

$\Leftrightarrow 2x^2 + 2y^2 + 2 < 10$

$\Leftrightarrow x^2 + y^2 + 1 < 5$

$\Leftrightarrow x^2 + y^2 < 4 \Leftrightarrow \sqrt{x^2 + y^2} < 2 \Leftrightarrow |z| < 2$

(21) A $\Leftrightarrow z = \frac{2\sqrt{3}}{3} + 2i$

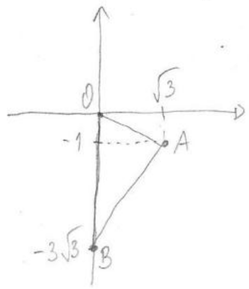
$\operatorname{tg}\theta = \frac{2}{\frac{2\sqrt{3}}{3}} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$ (C)

(22)

(22.1) $z = 2e^{i\frac{\pi}{6}} = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$

$w = \frac{(z-i)^4}{1+zi} = \frac{(\sqrt{3} + \cancel{i} - \cancel{i})^4}{1+(\sqrt{3}+i)i} = \frac{(\sqrt{3})^4}{1+\sqrt{3}i + i^2} = \frac{\left[(\sqrt{3})^2\right]^2}{\cancel{1}+\sqrt{3}i - \cancel{1}} = \frac{9}{\sqrt{3}i}$

$= \frac{9\sqrt{3}i}{(\sqrt{3}i)^2} = \frac{9\sqrt{3}i}{(\sqrt{3})^2 i^2} = \frac{9\sqrt{3}i}{3(-1)} = \frac{9\sqrt{3}i}{-3} = -3\sqrt{3}i$



$\bar{z} = \sqrt{3} - i$

$w = -3\sqrt{3}i$

$b = 3\sqrt{3}$ $A = \frac{b \times h}{2} = \frac{3\sqrt{3} \times \sqrt{3}}{2} = \frac{9}{2}$

$h = \sqrt{3}$

(22.2) $z^2 - 2\cos\alpha\, z + 1 = 0$

$\Leftrightarrow z = \frac{2\cos\alpha \pm \sqrt{(-2\cos\alpha)^2 - 4\times1\times1}}{2} = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2} =$

$= \frac{2\cos\alpha \pm \sqrt{4(1-\sin^2\alpha)-4}}{2} = \frac{2\cos\alpha \pm \sqrt{\cancel{4} - 4\sin^2\alpha - \cancel{4}}}{2} =$

$= \frac{2\cos\alpha \pm \sqrt{-4\sin^2\alpha}}{2} = \frac{2\cos\alpha \pm \sqrt{4\sin^2\alpha} \times \sqrt{-1}}{2} = \frac{2\cos\alpha \pm 2\sin\alpha\, i}{2}$

$= \cos\alpha \pm \sin\alpha\, i$ Soluções: $z_1 = \cos\alpha + \sin\alpha\, i$ ou $z_2 = \cos\alpha - \sin\alpha\, i$

ou $z_2 = \cos(-\alpha) + \sin(-\alpha)\, i$

Assim, $z_1 = e^{i\alpha} \vee z_2 = e^{i(-\alpha)}$