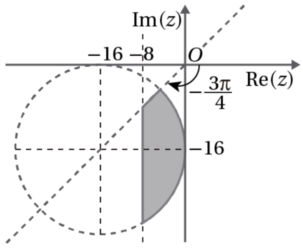
1.1. $|z_2| = |2-2\text{i}| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

Sendo $\text{Arg}(2-2\text{i}) = \theta$, tem-se $\tan\theta = \frac{-2}{2} \wedge \theta \in$ 4.º Q, ou seja, $\tan\theta = -1 \wedge \theta \in$ 4.º quadrante, então $\theta = -\frac{\pi}{4}$.

Portanto, $z_2 = 2\sqrt{2}\,\text{e}^{-\frac{\pi}{4}\text{i}}$.

$$z_1 = (2-2\text{i})^3 = \left(2\sqrt{2}\,\text{e}^{-\frac{\pi}{4}\text{i}}\right)^3 = \left(2\sqrt{2}\right)^3 \text{e}^{-\frac{3\pi}{4}\text{i}} = 16\sqrt{2}\,\text{e}^{-\frac{3\pi}{4}\text{i}}$$

1.2. $z_1 = 16\sqrt{2}\,\text{e}^{-\text{i}\frac{3\pi}{4}} = 16\sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\text{i}\right) = -16 - 16\text{i}$



2.1. $z_1^2 + bz_1 + c = 0 \Leftrightarrow \left(-1+\sqrt{3}\text{i}\right)^2 + b\left(-1+\sqrt{3}\text{i}\right) + c = 0 \Leftrightarrow$

$$\Leftrightarrow \left(-1+\sqrt{3}\text{i}\right)^2 + b\left(-1+\sqrt{3}\text{i}\right) + c = 0 \Leftrightarrow$$

$$\Leftrightarrow 1 - 2\sqrt{3}\text{i} + 3\text{i}^2 - b + \sqrt{3}b\text{i} + c = 0$$

$$\Leftrightarrow 1 - 2\sqrt{3}\text{i} - 3 - b + \sqrt{3}b\text{i} + c = 0 \Leftrightarrow$$

$$\Leftrightarrow -2 - b + c + \left(\sqrt{3}b - 2\sqrt{3}\right)\text{i} = 0$$

$$\Leftrightarrow -2 - b + c = 0 \wedge \sqrt{3}b - 2\sqrt{3} \Leftrightarrow c = 2 + b \wedge b = 2 \Leftrightarrow$$

$$\Leftrightarrow c = 4 \wedge b = 2 \Leftrightarrow$$

Portanto, $b = 2 \wedge c = 4$

2.2. $z_1 = -1 + \sqrt{3}\text{i}$

$|z_1| = \left|-1+\sqrt{3}\text{i}\right| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$

Se $\text{Arg}\left(-1+\sqrt{3}\text{i}\right) = \theta$:

$$\begin{cases} \tan\theta = \frac{\sqrt{3}}{-1} \Rightarrow \theta = \frac{2\pi}{3} \\ \theta \in 2.^\text{o}\, Q \end{cases}, \text{ logo } z_1 = 2^{\text{i}\frac{2\pi}{3}}.$$

$$\frac{z_1}{\overline{z_2}} = \frac{2\text{e}^{\text{i}\frac{2\pi}{3}}}{\text{e}^{-\text{i}\alpha}} = 2\text{e}^{\text{i}\left(\frac{2\pi}{3}+\alpha\right)}$$

$\frac{z_1}{\overline{z_2}}$ é um número imaginário puro de coeficiente negativo se

$$\frac{2\pi}{3} + \alpha = -\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow \alpha = -\frac{\pi}{2} - \frac{2\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \alpha = -\frac{7\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$$

Como $\alpha \in \left[-\pi, \pi\right[$, $\alpha = -\frac{7\pi}{6} + 2\pi = \frac{5\pi}{6}$.

3.1. Sejam $z_1 = \sqrt{2}\text{i} - \sqrt{2}$ e $z_2 = -\sqrt{3} - \text{i}$.

$|z_1| = \left|\sqrt{2}\text{i} - \sqrt{2}\right| = \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = \sqrt{4} = 2$

Sendo $\text{Arg}\left(-\sqrt{2} + \sqrt{2}\text{i}\right) = \theta$:

$$\begin{cases} \tan\theta = \frac{\sqrt{2}}{-\sqrt{2}} = -1 \Rightarrow \theta = \frac{3\pi}{4} \\ \theta \in 2.^\text{o}\, Q \end{cases}$$

Portanto, $z_1 = 2\text{e}^{\text{i}\frac{3\pi}{4}}$.

$|z_2| = \left|-\sqrt{3} - \text{i}\right| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = \sqrt{4} = 2$

Se $\beta = \text{Arg}\left(-\sqrt{3} - \text{i}\right)$:

$$\begin{cases} \tan\beta = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \beta = -\frac{5\pi}{6} \\ \beta \in 3.^\text{o}\, Q \end{cases}$$

Portanto, $z_2 = 2\text{e}^{-\text{i}\frac{5\pi}{6}}$.

$$w = \frac{z_1}{z_2} = \frac{2\text{e}^{\text{i}\frac{3\pi}{4}}}{2\text{e}^{-\text{i}\frac{5\pi}{6}}} = \text{e}^{\text{i}\left(\frac{3\pi}{4}+\frac{5\pi}{6}\right)} = \text{e}^{\text{i}\frac{19\pi}{12}} = \text{e}^{-\text{i}\frac{5\pi}{12}}$$

$\frac{19\pi}{12} - 2\pi = -\frac{5\pi}{12}$

3.2. $\text{e}^{-\text{i}\frac{5\pi}{12}} = \cos\left(-\frac{5\pi}{12}\right) + \text{i}\sin\left(-\frac{5\pi}{12}\right)$.

Por outro lado:

$$w = \frac{\sqrt{2}\text{i} - \sqrt{2}}{-\sqrt{3} - \text{i}} = \frac{\left(\sqrt{2}\text{i} - \sqrt{2}\right)\left(-\sqrt{3} + \text{i}\right)}{\left(-\sqrt{3} - \text{i}\right)\left(-\sqrt{3} + \text{i}\right)} = \frac{-\sqrt{6}\text{i} - \sqrt{2} + \sqrt{6} - \sqrt{2}\text{i}}{\left(-\sqrt{3}\right)^2 - \text{i}^2}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4} + \frac{-\sqrt{6} - \sqrt{2}}{4}\text{i}$$

então, vem:

$$\cos\left(-\frac{5\pi}{12}\right) + \text{i}\sin\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4} + \frac{-\sqrt{6} - \sqrt{2}}{4}\text{i}$$

$$\Leftrightarrow \cos\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4} \wedge \sin\left(-\frac{5\pi}{12}\right) = \frac{-\sqrt{6} - \sqrt{2}}{4}$$

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{6\pi}{12} - \frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{2} - \frac{5\pi}{12}\right) =$$

$$= \cos\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{6\pi}{12} - \frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{2} - \frac{5\pi}{12}\right) =$$

$$= -\sin\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Logo:

$$\cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{6} - \sqrt{2}}{4} = \frac{2\sqrt{6}}{4} = \frac{\sqrt{6}}{2}$$

3.3. $w^n = \left(\text{e}^{-\text{i}\frac{5\pi}{12}}\right)^n = \text{e}^{-\text{i}\frac{5\pi}{12}n}$

O afixo de w^n, no plano complexo, pertence ao semieixo positivo Ox se $-\frac{5\pi}{12}n = 2k\pi,\ k \in \mathbb{Z}$ e $n \in \mathbb{N}$.

$$-\frac{5\pi}{12}n = 2k\pi,\ k \in \mathbb{Z} \wedge n \in \mathbb{N} \Leftrightarrow$$

$$\Leftrightarrow n = -\frac{12}{5\pi} \times 2k\pi,\ k \in \mathbb{Z} \wedge n \in \mathbb{N} \Leftrightarrow$$

$$\Leftrightarrow n = -\frac{24k}{5},\ k \in \mathbb{Z} \wedge n \in \mathbb{N}$$

O menor valor de n natural, nestas condições, obtém-se quando $k = -5$, tendo-se $n = 24$.

4.1. $\displaystyle\sum_{k=0}^{30}\left[2\text{e}^{\text{i}\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)}\right] = 2\sum_{k=0}^{30}\left[\text{e}^{\text{i}\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)}\right] = 2\sum_{k=0}^{30}\left[\text{e}^{\text{i}\frac{\pi}{6}} \times \text{e}^{\text{i}\frac{k\pi}{3}}\right]$

$$= 2\text{e}^{\text{i}\frac{\pi}{6}} \times \sum_{k=0}^{30}\left(\text{e}^{\text{i}\frac{\pi}{3}}\right)^k = 2\text{e}^{\text{i}\frac{\pi}{6}} \times \left(\text{e}^0 + \text{e}^{\text{i}\frac{\pi}{3}} + \text{e}^{\text{i}\frac{2\pi}{3}} + \ldots + \text{e}^{\text{i}\frac{29\pi}{3}} + \text{e}^{\text{i}\frac{30\pi}{3}}\right)$$

$e^0, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, ..., e^{i\frac{29\pi}{3}}$ e $e^{i\frac{30\pi}{3}}$ são os primeiros 31 termos de uma progressão geométrica de razão $e^{i\frac{\pi}{3}}$ e cujo primeiro termo é e^0. Assim:

$$2e^{i\frac{\pi}{6}}\times\sum_{k=0}^{30}\left(e^{i\frac{\pi}{3}}\right)^k = 2e^{i\frac{\pi}{6}}\times\left[e^0\times\frac{1-\left(e^{i\frac{\pi}{3}}\right)^{31}}{1-e^{i\frac{\pi}{3}}}\right]=$$

$$=2e^{i\frac{\pi}{6}}\times\left(1\times\frac{1-e^{i\frac{31\pi}{3}}}{1-e^{i\frac{\pi}{3}}}\right)=$$

$\frac{31\pi}{3}=10\pi+\frac{\pi}{3}$

$$=2e^{i\frac{\pi}{6}}\times\left(1-\frac{1-e^{i\frac{\pi}{3}}}{1-e^{i\frac{\pi}{3}}}\right)=2e^{i\frac{\pi}{6}}\times1=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=$$

$$=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=\sqrt{3}+i$$

4.2. Tem-se:

$$z_2=\frac{i^{74}-2i^{11}}{i-3}\Leftrightarrow z_2=\frac{i^{18\times4+2}-2i^{2\times4+3}}{i-3}\Leftrightarrow z_2=\frac{i^2-2i^3}{i-3}$$

$$\Leftrightarrow z_2=\frac{-1-2(-i)}{i-3}\Leftrightarrow z_2=\frac{-1+2i}{-3+i}\Leftrightarrow$$

$$\Leftrightarrow z_2=\frac{(-1+2i)(-3-i)}{(-3+i)(-3-i)}\Leftrightarrow z_2=\frac{3+i-6i-2i^2}{(-3)^2-i^2}\Leftrightarrow$$

$$\Leftrightarrow z_2=\frac{5-5i}{10}\Leftrightarrow z_2=\frac{1}{2}-\frac{1}{2}i$$

$$|z_2|=\left|\frac{1}{2}-\frac{1}{2}i\right|=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{1}{4}}=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$$

Se $\text{Arg}\left(\frac{1}{2}-\frac{1}{2}i\right)=\theta$,

$$\begin{cases}\tan\theta=\frac{-\frac{1}{2}}{\frac{1}{2}}=-1\\ \theta\in 4.^{\text{o}}\ Q\end{cases}\Rightarrow\theta=-\frac{\pi}{4}\ \rightarrow \text{Assim, } z_2=\frac{\sqrt{2}}{2}e^{-i\frac{\pi}{4}}.$$

4.3. Seja $w=|w|e^{i\theta}$. Então, $\overline{w}=|w|e^{-i\theta}$.
Substituindo na equação:

$$w^3\times\overline{w}=(z_2)^4\Leftrightarrow\left(|w|e^{i\theta}\right)^3\times\left(|w|e^{-i\theta}\right)=\left(\frac{\sqrt{2}}{2}e^{-i\frac{\pi}{4}}\right)^4\Leftrightarrow$$

$$\Leftrightarrow|w|^3e^{3i\theta}\times|w|e^{-i\theta}=\left(\frac{\sqrt{2}}{2}\right)^4e^{-i\pi}\Leftrightarrow|w|^4e^{2i\theta}=\left(\frac{\sqrt{2}}{2}\right)^4e^{-i\pi}$$

$$\Leftrightarrow|w|^4=\left(\frac{\sqrt{2}}{2}\right)^4\wedge2\theta=-\pi+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$

$$\Leftrightarrow|w|=\frac{\sqrt{2}}{2}\wedge\theta=-\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$

$$\Leftrightarrow w=\frac{\sqrt{2}}{2}e^{-i\frac{\pi}{2}}\vee w=\frac{\sqrt{2}}{2}e^{i\frac{\pi}{2}}\Leftrightarrow$$

$$\Leftrightarrow w=-\frac{\sqrt{2}}{2}i\vee w=\frac{\sqrt{2}}{2}i$$

$$S=\left\{-\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2}i\right\}$$

5.1. $w=\frac{z_1}{z_2}\Leftrightarrow w=\frac{1+2ie^{i\frac{5\pi}{6}}}{1+i\sqrt{3}}$

$$z_1=1+2ie^{i\frac{5\pi}{6}}=1+2i\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=$$

$$=1+2i\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=1-\sqrt{3}i-1=-\sqrt{3}i=\sqrt{3}e^{-\frac{\pi}{2}i}$$

$$|z_2|=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$$

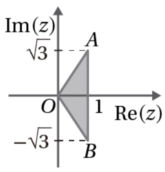
Se $\text{Arg}\left(1+i\sqrt{3}\right)=\theta$, tem-se:

$$\begin{cases}\tan\theta=\frac{\sqrt{3}}{1}\\ \theta\in 1.^{\text{o}}\ Q\end{cases}\Rightarrow\theta=\frac{\pi}{3}\ \text{. Assim, } z_2=2e^{i\frac{\pi}{3}}.$$

$$w=\frac{\sqrt{3}e^{-i\frac{\pi}{2}}}{2e^{i\frac{\pi}{3}}}=\frac{\sqrt{3}}{2}e^{-i\frac{\pi}{2}-i\frac{\pi}{3}}=\frac{\sqrt{3}}{2}e^{-i\frac{5\pi}{6}}$$

5.2. Tem-se que $z_2=1+i\sqrt{3}$, logo $z=\overline{z_2}=1-i\sqrt{3}$.
Representando o triângulo [*AOB*], no plano complexo:

$$P_{[AOB]}=\overline{OA}+\overline{OB}+\overline{AB}=$$
$$=|z_2|+|z|+2\sqrt{3}=$$
$$=2+2+2\sqrt{3}=$$
$$=4+2\sqrt{3}\ \text{u.c.}$$



Pág. 119

6.1. Tem-se que a multiplicação de um número complexo por $i=e^{i\frac{\pi}{2}}$ corresponde à rotação do seu afixo segundo um ângulo de amplitude $\frac{\pi}{2}$ radianos, no sentido positivo. Assim, e sendo *B*, *C* e *D* os afixos os números complexos z_2, z_3 e z_4, respetivamente, tem-se:

$$z_2=iz_1\Leftrightarrow z_2=i\left(\sqrt{2}+2\sqrt{2}i\right)\Leftrightarrow z_2=-2\sqrt{2}+\sqrt{2}i$$

$$z_3=i+z_2\Leftrightarrow z_3=i\left(-2\sqrt{2}+\sqrt{2}i\right)\Leftrightarrow z_3=-\sqrt{2}-2\sqrt{2}i$$

$$z_4=iz_3\Leftrightarrow z_4=i\left(-\sqrt{2}-2\sqrt{2}i\right)\Leftrightarrow z_4=2\sqrt{2}-\sqrt{2}i$$

Portanto, $B\left(-2\sqrt{2},\sqrt{2}\right)$, $C\left(-\sqrt{2},-2\sqrt{2}\right)$ e $D\left(2\sqrt{2},-\sqrt{2}\right)$.

6.2. A circunferência tem centro na origem do referencial e o seu raio é igual ao módulo de z_1.

$$|z_1|=\sqrt{\left(\sqrt{2}\right)^2+\left(2\sqrt{2}\right)^2}=\sqrt{2+8}=\sqrt{10},$$

Portanto, $|z|=\sqrt{10}$ é uma equação da circunferência circunscrita ao quadrado [*ABCD*].

6.3. Área pedida = Área do círculo de centro *O* e raio [*OA*] –
– Área do quadrado [*ABCD*]

Área do círculo de centro *O* e raio $[OA]=\pi\times\left(\sqrt{10}\right)^2=10\pi$.

Como [*ABCD*] é um quadrado de centro em *O*, o triângulo [*AOD*] é retângulo em *O*.
Pelo Teorema de Pitágoras:

$$\overline{AD}^2=\overline{OA}^2+\overline{OD}^2=\left(\sqrt{10}\right)^2+\left(\sqrt{10}\right)^2=20$$

Logo, a área do quadrado [*ABCD*] é igual a 20 u.a.e a área pedida é igual a (10π – 20) u.a.

7.1. Substituindo z por $x + y$i na condição: $\text{Re}(z - 3) \times \text{Im}(\text{i} \times \bar{z}) + |-2\text{i}| \leqslant 0$, vem:

$$x^2-3x+2=0 \Leftrightarrow x=\frac{3\pm\sqrt{9-8}}{2} \Leftrightarrow x=1 \vee x=2$$

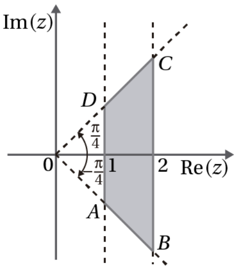
$$\text{Re}(x+y\text{i}-3)\times\text{Im}(\text{i}\times(x-y\text{i}))+2\le 0$$

$$\Leftrightarrow (x-3)\times\text{Im}(y+x\text{i})+2\le 0$$

$$\Leftrightarrow (x-3)x+2\le 0 \Leftrightarrow x^2-3x+2\le 0$$

$$\Leftrightarrow 1\le x\le 2$$

A região do plano complexo definida pela condição é o trapézio $[ABCD]$ representado na figura ao lado. Os pontos A, B, C e D têm as seguintes coordenadas: $A(1, -1)$, $B(2, -2)$, $C(2, 2)$ e $D(1, 1)$.



$$A_{[ABCD]} = \frac{\overline{BC}+\overline{AD}}{2}\times\text{altura} =$$

$$=\frac{4+2}{2}\times 1 = 3$$

Perímetro $[ABCD] = \overline{AD}+\overline{AB}+\overline{BC}+\overline{DC} = 2\overline{AB}+\overline{AD}+\overline{BC}$

$\overline{AB} = d(A\,,B) = \sqrt{(2-1)^2+(-2+1)^2} = \sqrt{2}$, logo

$P_{[ABCD]} = 2\sqrt2+2+4 = 6+2\sqrt2$.

7.2. $|z_1| = \left|-\frac34+\frac{3\sqrt3}{4}\text{i}\right| = \sqrt{\left(-\frac34\right)^2+\left(\frac{3\sqrt3}{4}\right)^2} = \sqrt{\frac{9}{16}+\frac{27}{16}} = \sqrt{\frac{36}{16}} = \frac32$

Se $\text{Arg}\left(-\frac34+\frac{3\sqrt3}{4}\text{i}\right)=\theta$, tem-se:

$$\begin{cases}\tan\theta = \dfrac{\frac{3\sqrt3}{4}}{-\frac34} \\ \theta\in 2.^{\text{o}}\,Q\end{cases} \Rightarrow \theta = \frac{2\pi}{3} \quad \rightarrow \text{Logo, } z_1 = \frac32 e^{\frac{2\pi}{3}\text{i}}.$$

Assim:

$$w = \frac{\bar{z}_2}{z_2+1}\times z_1 = \frac{2e^{-\text{i}\frac43\pi}}{2\left(\cos\frac{4\pi}{3}+\text{i}\sin\frac{4\pi}{3}\right)+1}\times\frac32 e^{\text{i}\frac{2\pi}{3}} =$$

$$= \frac{2e^{-\text{i}\frac43\pi}}{2\left(-\frac12-\frac{\sqrt3}{2}\text{i}\right)+1}\times\frac32 e^{\text{i}\frac{2\pi}{3}} = \frac{2e^{-\text{i}\frac43\pi}}{-1-\sqrt3\text{i}+1}\times\frac32 e^{\text{i}\frac{2\pi}{3}} =$$

$$= \frac{2e^{-\text{i}\pi\frac43}}{-\sqrt3\text{i}}\times\frac32 e^{\frac{2\pi}{3}\text{i}} = \frac{2e^{-\frac43\pi\text{i}}}{\sqrt3 e^{-\text{i}\frac{\pi}{2}}}\times\frac32 e^{\frac{2\pi}{3}\text{i}} = \frac{2}{\sqrt3}e^{-\text{i}\frac43\pi+\text{i}\frac{\pi}{2}}\times\frac32 e^{\text{i}\frac{2\pi}{3}}$$

$$= \frac{2\sqrt3}{3}e^{-\text{i}\frac{5\pi}{6}}\times\frac32 e^{\frac{2\pi}{3}\text{i}} = \frac{2\sqrt3}{3}\times\frac32 e^{-\frac{5\pi}{6}\text{i}+\frac{2\pi}{3}\text{i}} = \sqrt3 e^{-\text{i}\frac{\pi}{6}} =$$

$$= \sqrt3\left(\cos\frac{-\pi}{6}+\text{i}\sin\frac{-\pi}{6}\right) = \sqrt3\left(\frac{\sqrt3}{2}-\frac12\text{i}\right) = \frac32-\frac{\sqrt3}{2}\text{i}$$

Substituindo na condição $\text{Re}(z-3)\times\text{Im}(\text{i}\times z)+|-2\text{i}|\le 0$:

$$\text{Re}\left(\frac32-\frac{\sqrt3}{2}\text{i}-3\right)\times\text{Im}\left(\text{i}\times\left(\frac32+\frac{\sqrt3}{2}\text{i}\right)\right)+2\le 0 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac32-3\right)\times\text{Im}\left(\frac32\text{i}-\frac{\sqrt3}{2}\right)+2\le 0 \Leftrightarrow \left(-\frac32\right)\times\left(\frac32\right)+2\le 0$$

$$\Leftrightarrow -\frac94+2\le 0 \Leftrightarrow -\frac14\le 0 \quad \text{(Verdade)}$$

E como $\text{Arg}(w) = -\frac{\pi}{6}$ e $-\frac{\pi}{4}\le\text{Arg}(w)\le\frac{\pi}{4}$, podemos concluir que o afixo do número complexo w pertence ao trapézio.

8. $$\frac{(2-\text{i})^2+4e^{\frac{\pi}{2}\text{i}}+3\sqrt3\,\text{i}}{3e^{\text{i}\frac{\pi}{12}}}+8e^{i\frac{\pi}{4}} =$$

$$= \frac{3-4\text{i}+4\text{i}+3\sqrt3\,\text{i}}{3e^{\text{i}\frac{\pi}{12}}}+8\left(\frac{\sqrt2}{2}+\frac{\sqrt2}{2}\text{i}\right) =$$

$$= \frac{3+3\sqrt3\,\text{i}}{3e^{\text{i}\frac{\pi}{12}}}+4\sqrt2+4\sqrt2\,\text{i} = \frac{3\left(1+\sqrt3\,\text{i}\right)}{3e^{\text{i}\frac{\pi}{12}}}+4\sqrt2+4\sqrt2\,\text{i} =$$

$$= \frac{1+\sqrt3\,\text{i}}{e^{\text{i}\frac{\pi}{12}}}+4\sqrt2+4\sqrt2\,\text{i} = \frac{2e^{\text{i}\frac{\pi}{3}}}{e^{\text{i}\frac{\pi}{12}}}+4\sqrt2-4\sqrt2\,\text{i}$$

$z = 1+\sqrt3\,\text{i}$ e $\theta = \text{Arg}\,z$

$|z| = \sqrt{1+3} = 2$

$$\begin{cases}\tan\theta = \sqrt3 \\ \theta\in 1.^{\text{o}}\,Q\end{cases}\Rightarrow\theta = \frac{\pi}{3}$$

$$= 2e^{\text{i}\left(\frac{\pi}{3}-\frac{\pi}{12}\right)}+4\sqrt2+4\sqrt2\,\text{i} = 2e^{\text{i}\frac{\pi}{4}}+4\sqrt2-4\sqrt2\,\text{i}$$

$$= 2\left(\frac{\sqrt2}{2}+\frac{\sqrt2}{2}\text{i}\right)+4\sqrt2+4\sqrt2\,\text{i} = \sqrt2+\sqrt2\,\text{i}+4\sqrt2+4\sqrt2\,\text{i} =$$

$$= 5\sqrt2+5\sqrt2\,\text{i}$$

9.1. $$w_1 = \text{i}^{12n-9}\left(20\sqrt3-20\text{i}\right) = \text{i}^{12n}\times\text{i}^{-9}\times\left(20\sqrt3-20\text{i}\right) =$$

$$= \left(\text{i}^4\right)^{3n}\times\text{i}^{-9+4\times3}\times\left(20\sqrt3-20\text{i}\right) = 1\times\text{i}^3\times\left(20\sqrt3-20\text{i}\right) =$$

$$= -\text{i}\times\left(20\sqrt3-20\text{i}\right) = -20-20\sqrt3\,\text{i}$$

Assim:

$$|w_1| = \left|-20-20\sqrt3\,\text{i}\right| = \sqrt{(-20)^2+\left(-20\sqrt3\right)^2} = \sqrt{400+1200} = 40$$

Se $\theta = \text{Arg}\left(-20-2\sqrt3\,\text{i}\right)$:

$$\begin{cases}\tan\theta = \dfrac{-20\sqrt3}{-20} \\ \theta\in 3.^{\text{o}}\,Q\end{cases}\Rightarrow\theta = -\frac{2\pi}{3}\text{, logo } w_1 = 40e^{-\text{i}\frac{2\pi}{3}}.$$

Por outro lado, w_2 é uma raiz cúbica de w_1 quando e apenas quando $w_1 = (w_2)^3$.

$$(w_2)^3 = \left(2\sqrt[3]{5}\,e^{\text{i}\frac{10\pi}{9}}\right)^3 = \left(2\sqrt[3]{5}\right)^3 e^{\text{i}\frac{30\pi}{9}} = 8\times 5e^{\text{i}\frac{10\pi}{3}} = 40e^{-\text{i}\frac{2\pi}{3}} =$$

$$= w_1$$

$$\frac{10\pi}{3} = -\frac{2\pi}{3}+4\pi$$

Como $w_1 = (w_2)^3$, podemos concluir que de facto w_2 é uma única raiz cúbica de w_1, como queríamos mostrar.

9.2. Tem-se que o afixo de w_1 é o ponto de coordenadas $\left(-20\,,-20\sqrt3\right)$.

O raio da circunferência é igual à distância do afixo de w_1 à origem do referencial, ou seja, é igual ao módulo de w_1.

Tem-se que $|z - w_1| = 40$ é uma equação da circunferência

Por isso, uma condição que define a parte da circunferência que está contida no segundo quadrante (eixos não incluídos) é, por exemplo, $\left|z-\left(-20-20\sqrt3\,\text{i}\right)\right| = 40 \wedge \text{Im}(z) > 0$.

Pág. 120

10.1. Determinemos as raízes de índice 5 de 32i.

$$\sqrt[5]{32\text{i}} = \sqrt[5]{32e^{\text{i}\frac{\pi}{2}}} = \sqrt[5]{32}\,e^{\text{i}\left(\frac{\frac{\pi}{2}}{5}+\frac{2k\pi}{5}\right)}, k = \{0\,,1\,,2\,,3\,,4\} =$$

$$= 2e^{\text{i}\frac{\pi+4k\pi}{10}}, k = \{0\,,1\,,2\,,3\,,4\}$$

As raízes de índice 5 de 32i são $2e^{\pi\frac{\pi}{10}}$, $2e^{i\frac{\pi}{2}}$, $2e^{i\frac{9\pi}{10}}$, $2e^{i\frac{13\pi}{10}}$ e $2e^{i\frac{17\pi}{10}}$. Das cinco raízes, a única que tem argumento pertencente ao terceiro quadrante é $2e^{\frac{13\pi}{10}i}$. Logo, o ponto *A* é o afixo deste número complexo. Por outro lado, a reta *r* tem inclinação igual a 150º, ou seja, $\frac{5\pi}{6}$ radianos.

$|z| = 2$ é uma equação da circunferência.

Assim, a região sombreada, incluindo a fronteira pode ser definida pela condição: $|z| < 2 \wedge \frac{5\pi}{6} \leq \operatorname{Arg}(z) \leq \frac{13\pi}{10}$

10.2. $w = \dfrac{\left(\sqrt{3}+i\right)^2}{\dfrac{\left(2e^{i\frac{\pi}{3}}\right)^3 + 2}{-i}} = \dfrac{-i\left(3+2\sqrt{3}i+i^2\right)}{2^3 e^{i\pi}+2} = \dfrac{-i\left(3+2\sqrt{3}i-1\right)}{-8+2} =$

$$= \frac{-i\left(2+2\sqrt{3}i\right)}{-6} = \frac{2\sqrt{3}-2i}{-6} = -\frac{\sqrt{3}}{3} + \frac{1}{3}i$$

$|w| = \sqrt{\frac{3}{9}+\frac{1}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$. Se $\theta = \operatorname{Arg} w$:

$$\begin{cases} \tan\theta = \dfrac{1}{-\sqrt{3}} = -\dfrac{\sqrt{3}}{3} \Rightarrow \theta = \dfrac{5\pi}{6} \\ \theta \in 2.^{\text{o}}\, Q \end{cases} \rightarrow \quad w = \frac{2}{3}e^{i\frac{5\pi}{6}}$$

O afixo de *w* está no interior da circunferência, pois $|w| = \frac{2}{3}$ e $\frac{2}{3} < 2$. A inclinação da reta *r* é $\frac{5\pi}{6}$ e $\operatorname{Arg}(w) = \frac{5\pi}{6}$, portanto, o afixo de *w* está sobre a reta *r*.

11.1. $|z_1| = 1$ e $z_2 \times \overline{z_2} = |z_2|^2 = \left(3\sqrt{2}\right)^2 = 18$

$$\frac{\sqrt{2}|z_1|}{z_1} - \frac{z_2\times\overline{z_2}}{\sqrt{2}i} = \frac{\sqrt{2}\times 1}{e^{i\frac{2\pi}{3}}} - \frac{18}{\sqrt{2}i} = \sqrt{2}e^{-i\frac{2\pi}{3}} - \frac{18\times\left(-\sqrt{2}i\right)}{\sqrt{2}i\times\left(-\sqrt{2}i\right)}$$

$$= \sqrt{2}\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right) + \frac{18\sqrt{2}i}{2} =$$

$$= \sqrt{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + 9\sqrt{2}i = -\frac{\sqrt{2}}{2} + \left(9\sqrt{2} - \frac{\sqrt{6}}{2}\right)i$$

11.2. $\left|z - z^3\right| = \left|z\left(1-z^2\right)\right| = |z|\left|1-z^2\right| = 1\left|1-e^{2i\alpha}\right| =$

$$z = e^{i\alpha} \qquad z^2 = e^{i2\alpha} \qquad |z| = 1$$

$$= \left|1-e^{2i\alpha}\right| = \left|1-\left(\cos(2\alpha)+i\sin(2\alpha)\right)\right| =$$

$$= \left|\left(1-\cos(2\alpha)\right)+i\sin(2\alpha)\right| =$$

$$= \sqrt{\left(1-\cos(2\alpha)\right)^2 + \left(\sin(2\alpha)\right)^2} =$$

$$= \sqrt{1-2\cos(2\alpha)+\cos^2(2\alpha)+\sin^2(2\alpha)} =$$

$$= \sqrt{1-2\cos(2\alpha)+1} = \sqrt{2-2\cos(2\alpha)} =$$

$$= \sqrt{2-2\left(\cos^2\alpha - \sin^2\alpha\right)} = \sqrt{2-2\left(1-\sin^2\alpha-\sin^2\alpha\right)} =$$

$$= \sqrt{2-2+4\sin^2\alpha} = \sqrt{4\sin^2\alpha} = 2|\sin\alpha|$$

12.1. $z_1 + z_2^3 = \sqrt{2}e^{-i\frac{3\pi}{4}} + (3+4i)^3 =$

$$= \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) + (3+4i)^2(3+4i) =$$

$$= \sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) + \left(9+24i+16i^2\right)(3+4i) =$$

$$= -1 - i + (-7+24i)(3+4i) =$$

$$= -1 - i + \left(-21-28i+72i+96i^2\right) =$$

$$= -1 - i + (-117+44i) = -118+43i$$

12.2. $z^3 = z_1^2 \Leftrightarrow z^3 = \left(\sqrt{2}e^{-i\frac{3\pi}{4}}\right)^2 \Leftrightarrow z^3 = \left(\sqrt{2}\right)^2 e^{-i\frac{3\pi}{2}} \Leftrightarrow z^3 = 2e^{-i\frac{3\pi}{2}}$

$$\Leftrightarrow z = \sqrt[3]{2e^{-i\frac{3\pi}{2}}} \Leftrightarrow z = \sqrt[3]{2}e^{i\left(-\frac{3\pi}{6}+\frac{2k\pi}{3}\right)},\ k = \{0,1,2\}$$

$$\Leftrightarrow z = \sqrt[3]{2}e^{-i\frac{\pi}{2}} \vee z = \sqrt[3]{2}e^{i\frac{\pi}{6}} \vee z = \sqrt[3]{2}e^{i\frac{5\pi}{6}}$$

13.1. $w = \dfrac{z_2 \times i^{4n+3} - b}{\sqrt{2}e^{-i\frac{\pi}{4}}} = \dfrac{\left(1-\sqrt{2}i\right)\times\left(i^4\right)^n \times i^3 - b}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)} =$

$$= \frac{\left(1-\sqrt{2}i\right)\times i^3 - b}{\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)} = \frac{\left(1-\sqrt{2}i\right)(-i) - b}{1-i} =$$

$$= \frac{-i-\sqrt{2}-b}{1-i} = \frac{-\sqrt{2}-b-i}{1-i} = \frac{\sqrt{2}+b+i}{-1+i} =$$

$$= \frac{\left(\sqrt{2}+b+i\right)(-1-i)}{(-1+i)(-1-i)} = \frac{-\sqrt{2}-\sqrt{2}i-b-bi-i-i^2}{(-1)^2-i^2} =$$

$$= \frac{-\sqrt{2}-\sqrt{2}i-b-bi-i+1}{2} = \frac{-\sqrt{2}-b+1+\left(-\sqrt{2}-b-1\right)i}{2}$$

$$= \frac{-\sqrt{2}-b+1}{2} + \frac{-\sqrt{2}-b-1}{2}i$$

w é um número imaginário puro. Logo:

$$\frac{-\sqrt{2}+b+1}{2} \wedge \frac{-\sqrt{2}-b-1}{2} \neq 0 \Leftrightarrow -\sqrt{2}-b+1 = 0 \wedge -\sqrt{2}-b-1 \neq 0$$

$$\Leftrightarrow -\sqrt{2}+1 = b \wedge -\sqrt{2}-1 \neq b \Leftrightarrow b = -\sqrt{2}+1$$

Portanto, $b = -\sqrt{2}+1$.

13.2. $\cos x - \sqrt{3}\sin x + yi = z_1 \Leftrightarrow \cos x - \sqrt{3}\sin + yi = 1 - \sqrt{2}i \Leftrightarrow$

$$\Leftrightarrow \cos x - \sqrt{3}\sin x = 1 \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \frac{1}{2} \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{3}\right)\cos x - \sin\left(\frac{\pi}{3}\right)\sin x = \cos\left(\frac{\pi}{3}\right) \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{3}+x\right) = \cos\left(\frac{\pi}{3}\right) \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{\pi}{3}+x = \frac{\pi}{3}+2k\pi \vee \frac{\pi}{3}+x = -\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}\right) \wedge y = -\sqrt{2} \Leftrightarrow$$

$$\Leftrightarrow \left(x = 2k\pi \vee x = -\frac{2\pi}{3}+2k\pi,\ k\in\mathbb{Z}\right) \wedge y = -\sqrt{2}$$

Pág. 121

14.1. $|z - 3| = |z - (3 + 0i)|$ representa a distância do afixo de *z* ao ponto de coordenadas (3, 0) e $|z + 3| = |z - (-3 + 0i)|$ representa a distância do afixo de *z* ao ponto de coordenadas (– 3, 0). Se $F_1(3, 0)$ e $F_2(-3, 0)$, $|z-3| + |z+3| = 10$ define o conjunto de pontos *z*, do plano complexo, tais que a soma das distâncias de cada um deles a dois pontos fixos do plano complexo, F_1 e F_2, é constante e maior que a distância $\overline{F_1F_2}$.

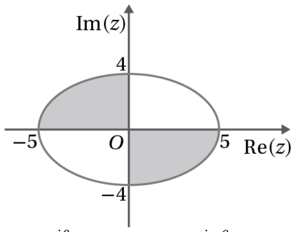
Conclui-se que a condição $|z - 3| + |z + 3| = 10$ define a elipse, de centro na origem do referencial, focos F_1 e F_2, de coordenadas (3, 0) e (– 3, 0), respetivamente, e eixo maior igual a 10. Por outro lado, os focos situam-se no eixo Ox, pelo que o eixo maior é igual a $2a$, ou seja, $2a = 10$ e, portanto, $a = 5$. Como $a > b$, $a^2 = b^2 + c^2$, $a^2 = b^2 + c^2$, logo:

$5^2 = b^2 + 3^2 \Leftrightarrow b^2 = 16$, então $b = 4$.

Focos: $F_1(3, 0)$ e $F_2(-3, 0)$

Vértices: $A(-5, 0)$, $B(5, 0)$, $C(0, -4)$ e $D(0, 4)$

14.2.



15. $w = e^{i\theta}$, então $w^n = e^{i\,n\theta}$ e $w^{-n} = e^{-i\,n\theta}$, portanto:

$$\frac{w^n - w^{-n}}{2i} = \frac{e^{in\theta} - e^{-in\theta}}{2i} =$$

$$= \frac{\cos(n\theta) + i\sin(n\theta) - [\cos(-n\theta) + i\sin 0 n\theta]}{2i} =$$

$$= \frac{\cos(n\theta) + i\sin(n\theta) - \cos(-n\theta) - i\sin(-n\theta)}{2i} =$$

$$= \frac{\cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta)}{2} =$$

$$= \frac{i\sin(n\theta) + i\sin(n\theta)}{2i} = \frac{2i\sin(n\theta)}{2i} = \sin(n\theta)$$

16.1. $$z_1 = \frac{\cos\alpha + i\sin(-\alpha)}{\sin(\alpha) + i\sin(-\alpha)} = \frac{\cos(-\alpha) + i\sin(-\alpha)}{\sin(\alpha) + i\cos(\alpha)} =$$

$$= \frac{e^{-\alpha i}}{\cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{e^{-i\alpha}}{e^{i\left(\frac{\pi}{2} - \alpha\right)}} = e^{i\left[-\alpha - \left(\frac{\pi}{2} - \alpha\right)\right]} =$$

$$= e^{-i\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i\times(-1) = -i$$

Portanto, z_1, é um imaginário puro de coeficiente negativo negativo, já que $\text{Re}(z_1) = 0 \wedge \text{Im}(z_1) < 0$.

16.2. $$z_2 = \frac{\left[i^{4n+1} \times \left(e^{i\frac{\pi}{5}}\right)^5 - 1\right]^2}{2i^{10}} = \frac{\left(i^{4n} \times i \times e^{i\pi} - 1\right)^2}{2i^{2\times4+10} \times i^2} = \frac{\left(i \times e^{i\pi} - 1\right)^2}{2 \times \left(i^4\right)^2 \times i^2}$$

$$= \frac{(-i-1)^2}{2 \times 1 \times (-1)} = \frac{i^2 + 2i + 1}{-2} = \frac{2i}{-2} = -i$$, logo $z_2 = z_1$.

17.1. $$w_3 = \frac{i}{1-i} - i^{24n+2} = \frac{i}{1-i} - \left(i^4\right)^{6n} \times i^2 = \frac{i}{1-i} - i^2 = \frac{i}{1-i} + 1 =$$

$$= \frac{i + 1 - i}{1 - i} = \frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} =$$

$$= \frac{1}{2}(1+i) = \frac{1}{2} \times \sqrt{2}\, e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}}$$

$z = 1 + i$ e $\theta = \text{Arg}(z)$

$|z| = \sqrt{1+1} = \sqrt{2}$

$\begin{cases} \tan\theta = 1 \\ \theta \in 1.^\text{o}\ Q \end{cases} \Rightarrow \theta = \frac{\pi}{4}$

$z = \sqrt{2}\, e^{i\frac{\pi}{4}}$

17.2. $$\left(i^3 w_2\right)^n + 1 = 0 \Leftrightarrow (-i\,w_2)^n = -1 \Leftrightarrow \left(-i\,e^{i\frac{5\pi}{6}}\right)^n = -1$$

$$\Leftrightarrow \left(e^{-i\frac{\pi}{2}} e^{i\frac{5\pi}{6}}\right)^n = -1 \Leftrightarrow \left(e^{i\left(-\frac{\pi}{2} + \frac{5\pi}{6}\right)}\right)^n = -1 \Leftrightarrow \left(e^{i\frac{\pi}{3}}\right)^n = -1$$

$$\Leftrightarrow e^{i\frac{\pi}{3}n} = e^{-i\pi} \Leftrightarrow \frac{\pi}{3}n = \pi + 2k\pi, k \in \mathbb{Z} \Leftrightarrow n = 3 + 6k, k \in \mathbb{Z}$$

O menor valor de $n \in \mathbb{N}$, obtém-se para $k = 0$, ou seja: $n = 3 + 6 \times 0 = 3$

Logo, $n = 3$.

17.3. $$z = \frac{\overline{w_1}}{w_1 + 1} = \frac{2e^{-\frac{2\pi}{3}i}}{2e^{i\frac{2\pi}{3}} + 1} = \frac{2e^{-i\frac{2\pi}{3}}}{2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + 1} =$$

$$= \frac{2e^{-i\frac{2\pi}{3}}}{2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1} = \frac{2e^{-i\frac{2\pi}{3}}}{-1 + \sqrt{3}i + 1} = \frac{2e^{-i\frac{2\pi}{3}}}{\sqrt{3}i} = \frac{2e^{-i\frac{2\pi}{3}}}{\sqrt{3}e^{i\frac{\pi}{2}}} =$$

$$= \frac{2}{\sqrt{3}} e^{i\left(-\frac{2\pi}{3} - \frac{\pi}{2}\right)} = \frac{2\sqrt{3}}{3} e^{-\frac{7\pi}{6}i}$$

18.1. $$|z| = \sqrt{4^2 + \left(4\sqrt{3}\right)^2} = \sqrt{16 + 48} = \sqrt{64} = 8$$

Sendo $\text{Arg}\, z = \theta$:

$$\begin{cases} \tan\theta = \frac{4\sqrt{3}}{4} \Rightarrow \theta = \frac{\pi}{3} \\ \theta \in 1.^\text{o}\ Q \end{cases} \rightarrow z = 8e^{i\frac{\pi}{3}}$$

$$\sqrt[4]{z} = \sqrt[4]{8e^{i\frac{\pi}{3}}} = \sqrt[4]{8}\, e^{i\left(\frac{\frac{\pi}{3}}{4} + \frac{2k\pi}{4}\right)} = \sqrt[4]{8}\, e^{i\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)}, i = 0, 1, 2, 3$$

Para $k = 0: z_0 = \sqrt[4]{8}\, e^{i\frac{\pi}{12}}$; para $k = 1: z_1 = \sqrt[4]{8}\, e^{i\frac{7\pi}{12}}$;

para $k = 2: z_2 \sqrt[4]{8}\, e^{i\frac{13\pi}{12}}$; para $k = 3$: $z_3 = \sqrt[4]{8}\, e^{i\frac{19\pi}{12}}$

18.2. Os afixos das raízes quartas de z são vértices de um quadrado. O quadrado está inscrito numa circunferência de raio $\sqrt[4]{8}$ de centro na origem do referencial, $|z| = \sqrt[4]{8}$ é uma equação dessa circunferência.

19.1. $$z^4 + zz_1 = 0 \Leftrightarrow z^4 + z\left(\sqrt{3} + i\right) = 0$$

$$\Leftrightarrow z\left(z^3 + \sqrt{3} + i\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow z = 0 \vee z^3 + \sqrt{3} + i = 0 \Leftrightarrow$$

$$\Leftrightarrow z = 0 \vee z^3 = -\sqrt{3} - i \Leftrightarrow$$

$z = -\sqrt{3} - i$ e

θ um argumento de z

$|z| = \sqrt{3 + 1} = 2$

$\begin{cases} \tan\theta = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \\ \theta \in 4.^\text{o}\ Q \end{cases}$

$\Rightarrow \frac{7\pi}{6}$ é um argumento de z

$$\Leftrightarrow z = 0 \vee z = \sqrt[3]{-\sqrt{3} - i} \Leftrightarrow z = 0 \vee z = \sqrt[3]{2e^{i\frac{7\pi}{6}}} \Leftrightarrow$$

$$\Leftrightarrow z = 0 \vee x = \sqrt[3]{2}\, e^{i\left(\frac{7\pi}{18} + \frac{2k\pi}{3}\right)}, k = 0, 1, 2$$

$$\Leftrightarrow z = 0 \vee z = \sqrt[3]{2}\, e^{i\frac{7\pi}{18}} \vee z = \sqrt[3]{2}\, e^{i\frac{19\pi}{18}} \vee z = \sqrt[3]{2}\, e^{i\frac{31\pi}{18}}$$

19.2. $$|z_2 + z_3|^2 = \left|e^{i\frac{\pi}{5}} + 3 + i\right|^2 = \left|\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} + 3 + i\right|^2 =$$

$$= \left|\left(\cos\frac{\pi}{5} + 3\right) + \left(\sin\frac{\pi}{5} + 1\right)i\right|^2 =$$

$$= \left[\sqrt{\left(\cos\frac{\pi}{5} + 3\right)^2 + \left(\sin\frac{\pi}{5} + 1\right)^2}\right]^2 =$$

$$= \left(\cos\frac{\pi}{5} + 3\right)^2 + \left(\sin\frac{\pi}{5} + 1\right)^2 =$$

$$= \cos^2\frac{\pi}{5} + 6\cos\frac{\pi}{5} + 9 + \sin^2\frac{\pi}{5} + 2\sin\frac{\pi}{5} + 1 =$$

$$= \cos^2\frac{\pi}{5} + \sin^2\frac{\pi}{5} + 6\cos\frac{\pi}{5} + 2\sin\frac{\pi}{5} + 10 =$$

$$= 1 + 6\cos\frac{\pi}{5} + 2\sin\frac{\pi}{5} + 10 = 11 + 6\cos\frac{\pi}{5} + 2\sin\frac{\pi}{5}$$

19.3. $w=\dfrac{\text{i}-(z_2)^{10}}{z_1-\text{i}^5}=\dfrac{\text{i}-\left(\text{e}^{\text{i}\frac{\pi}{5}}\right)^{10}}{\sqrt{3}+\text{i}-\text{i}^4\times\text{i}}==\dfrac{\text{i}-\text{e}^{\text{i}2\pi}}{\sqrt{3}+\text{i}-\text{i}}$

$=\dfrac{\text{i}-1}{\sqrt{3}}=\dfrac{\sqrt{2}\,\text{e}^{\text{i}\frac{3\pi}{4}}}{\sqrt{3}\,\text{e}^{\text{i}\times 0}}=\dfrac{\sqrt{2}}{\sqrt{3}}\text{e}^{\text{i}\left(\frac{3\pi}{4}-0\right)}=\dfrac{\sqrt{6}}{3}\text{e}^{\text{i}\frac{3\pi}{4}}$

$z=-1+\text{i}$ e $\theta=\text{Arg}(z)$

$|z|=\sqrt{1+1}=\sqrt{2}$

$\begin{cases}\tan\theta=-1\\ \theta\in 2.^{\text{o}}\,Q\end{cases}\Rightarrow\theta=\dfrac{3\pi}{4}$

$z=\sqrt{2}\,\text{e}^{\text{i}\frac{3\pi}{4}}$

19.4. $z_1=\sqrt{3}+\text{i}=2\text{e}^{\text{i}\frac{\pi}{6}}$ e $z_2=\text{e}^{\text{i}\frac{\pi}{5}}$

$(z_1\times z_2)^n=\left(2\text{e}^{\text{i}\frac{\pi}{6}}\times\text{e}^{\text{i}\frac{\pi}{5}}\right)^n=\left(2\text{e}^{\text{i}\left(\frac{\pi}{6}+\frac{\pi}{5}\right)}\right)$

$=\left(2\text{e}^{\text{i}\frac{11\pi}{30}}\right)^n=2^n\,\text{e}^{\text{i}\frac{11\pi}{30}n}$

$z=\sqrt{3}+\text{i}$ e $\theta=\text{Arg}(z)$

$|z|=\sqrt{3+1}=2$

$\begin{cases}\tan\theta=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}\Rightarrow\theta=\dfrac{\pi}{6}\\ \theta\in 1.^{\text{o}}\,Q\end{cases}$

$\sqrt{3}+\text{i}=2\text{e}^{\text{i}\frac{\pi}{6}}$

$\left(2^n\times\text{i}\dfrac{11\pi}{30}n\right)\in\mathbb{R}^+\Leftrightarrow\dfrac{11\pi}{30}n=2k\pi,\,k\in\mathbb{Z}\Leftrightarrow$

$\Leftrightarrow n=\dfrac{30}{11\pi}\times 2k\pi,\,k\in\mathbb{Z}\Leftrightarrow n=\dfrac{60}{11}k,\,k\in\mathbb{Z}$

Portanto, o menor valor de n natural para o qual $(z_1\times z_2)^n$ é um número real positivo, $n=60$, é obtido para $k=11$.

Pág. 122

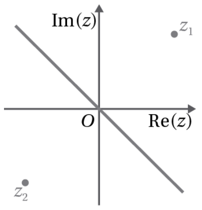
20. $\left|\dfrac{z-2-2\text{i}}{z+2+2\text{i}}\right|\Leftrightarrow\dfrac{|z-2-2\text{i}|}{|z+2+2\text{i}|}=1\wedge z+2+2\text{i}\neq 0\Leftrightarrow$

$\Leftrightarrow|z-2-2\text{i}|=|z+2+2\text{i}|\wedge z\neq-2-2\text{i}\Leftrightarrow$

$\Leftrightarrow|z-(2+2\text{i})|=|z-(-2-2\text{i})|\wedge z\neq-2-2\text{i}$

A condição na variável complexa $|z-(2+2\text{i})|=|z-(-2-2\text{i})|$ define a mediatriz do segmento de reta $[z_1z_2]$, com $z_1=2+2\text{i}$ e $z_2=-2-2\text{i}$.

Ao lado representa-se esse conjunto de pontos no plano complexo.



21.1. Pretende-se determinar $w=x+\text{i}y$, tal que $w^2=7+24\text{i}$.

$(z+\text{i}\,y)^2=7+24\text{i}\Leftrightarrow x^2+2xy\text{i}+\text{i}^2y^2=7+24\text{i}$

$\Leftrightarrow(x^2-y^2)+2xy\text{i}=7+24\text{i}\quad\Leftrightarrow\begin{cases}x^2-y^2=7\\2xy=24\end{cases}\Leftrightarrow$

$\Leftrightarrow\begin{cases}x^2-\left(\dfrac{12}{x}\right)^2=7\\y=\dfrac{12}{x}\end{cases}\Leftrightarrow\begin{cases}x^2-\dfrac{144}{x^2}=7\\y=\dfrac{12}{x}\end{cases}\Leftrightarrow\begin{cases}x^4-7x^2-144=0\\y=\dfrac{12}{x}\end{cases}$

$\Leftrightarrow\begin{cases}x^2=\dfrac{7\pm\sqrt{(-7)^2-4\times(-144)}}{2}\\y=\dfrac{12}{x}\end{cases}\Leftrightarrow\begin{cases}x^2=\dfrac{7\pm 25}{2}\\y=\dfrac{12}{x}\end{cases}\Leftrightarrow$

$\Leftrightarrow\begin{cases}x^2=16\vee x^2=-9\\y=\dfrac{12}{x}\end{cases}\underset{x\in\mathbb{R}}{\Leftrightarrow}\begin{cases}x^2=16\vee x\in\varnothing\\y=\dfrac{12}{x}\end{cases}\Leftrightarrow\begin{cases}x^2=16\\y=\dfrac{12}{x}\end{cases}$

$\Leftrightarrow\begin{cases}x=4\\y=3\end{cases}\vee\begin{cases}x=-4\\y=-3\end{cases}$

As raízes quadradas de $z_1=7+24\text{i}$ são $4+3\text{i}$ e $-4-3\text{i}$.

21.2. $z^3-z^2\text{i}+16(z-\text{i})=0\Leftrightarrow z^3-z^2+16z-16\text{i}=0$

Como $z_2=\text{i}$ é a solução da equação, podemos recorrer à regra de Ruffini:

	1	– i	16	– 16i
i		i	0	16i
	1	0	16	0

$z^3-z^2\text{i}+16z-16\text{i}=0\Leftrightarrow(z-\text{i})(z^2+16)=0\Leftrightarrow$

$\Leftrightarrow z-\text{i}=0\vee z^2+16=0\Leftrightarrow z=\text{i}\vee z^2=-16\Leftrightarrow$

$\Leftrightarrow z=\text{i}\vee z=-\sqrt{-16}\vee z=\sqrt{-16}\Leftrightarrow$

$\Leftrightarrow z=\text{i}\vee z=-4\text{i}\vee z=4\text{i}$

22.1. Como o pentágono é regular, os arcos de circunferência compreendidos entre dois vértices consecutivos são iguais.

Cada um deles tem amplitude igual a $\dfrac{2\pi}{5}$.

Por outro lado, o afixo do número complexo $z_1=-3\text{i}$ é o ponto *D*, logo $z_1=3\text{e}^{-\text{i}\frac{\pi}{2}}$.

Assim, o número complexo z_2 cuja imagem geométrica é o ponto *B* é $z_2=3\text{e}^{\text{i}\left(-\frac{\pi}{2}+3\times\frac{2\pi}{5}\right)}=3\text{e}^{\text{i}\frac{7\pi}{10}}$.

22.2. Sejam z_3 e z_4 os números complexos cujos afixos são os pontos *A* e *E*, respetivamente.

$z_3=3\text{e}^{\text{i}\left(-\frac{\pi}{2}+2\times\frac{2\pi}{5}\right)}=3\text{e}^{\text{i}\frac{3\pi}{10}}$ e $z_4=3\text{e}^{\text{i}\left(-\frac{\pi}{2}+\frac{2\pi}{5}\right)}=3\text{e}^{-\text{i}\frac{\pi}{10}}$

A circunferência tem centro na origem do referencial e raio igual a $|z_1|=3$, logo, pode ser definida pela equação $|z|=3$.

Portanto, uma condição que define a região sombreada (exluindo fronteira) é: $|z|<3\wedge-\dfrac{\pi}{10}<\text{Arg}(z)<\dfrac{3\pi}{10}$

23.1. Seja $z_1=|z_1|\text{e}^{\text{i}\theta}$, com $\theta\in\left]\dfrac{\pi}{2},\pi\right[$. Então, $z^3=|z_0|^3\,\text{e}^{\text{i}3\theta}$.

Se $\dfrac{\pi}{2}<\theta<\pi$, então $\dfrac{3\pi}{2}<3\theta<3\pi$. Se $3\theta\in\left]\dfrac{3\pi}{2},3\pi\right[$, o afixo de z^3 não pode pertencer ao terceiro quadrante.

23.2. $z_3\times\overline{z_3}=|z_3|^2=(2\sqrt{3})^2=12$

$\dfrac{z_3\times\overline{z_3}}{8}=\dfrac{12}{8}=\dfrac{3}{2}$

Como $z_2=|z_2|\text{e}^{\text{i}\frac{\pi}{3}}$, $\dfrac{z_2}{|z_2|}=\dfrac{|z_2|\text{e}^{\text{i}\frac{\pi}{3}}}{|z_2|}=\text{e}^{\text{i}\frac{\pi}{3}}$ e

$\left(\dfrac{z_2}{|z_2|}\right)^9=\left(\text{e}^{\text{i}\frac{\pi}{3}}\right)^9=\left(\text{e}^{\text{i}\frac{9\pi}{3}}\right)=\text{e}^{\text{i}3\pi}=\text{e}^{\text{i}\pi}=-1$

Portanto, $\dfrac{z_3\times\overline{z_3}}{8}+\left(\dfrac{z_2}{|z_2|}\right)^9=\dfrac{3}{2}+(-1)=\dfrac{1}{2}$.

24.1. $(z_1)^3=z\text{i}+2z\Leftrightarrow(1+2\text{i})^3=z\text{i}+2z$

$\Leftrightarrow(1+2\text{i}^2)(1+2\text{i})=z(\text{i}+2)\Leftrightarrow$

$\Leftrightarrow(1+4\text{i}+4\text{i}^2)(1+2\text{i})=z(\text{i}+2)$

$\Leftrightarrow(-3+4\text{i})(1+2\text{i})=z(\text{i}+2)\Leftrightarrow$

$\Leftrightarrow-3-6\text{i}+4\text{i}+8\text{i}^2=z(\text{i}+2)\Leftrightarrow$

$\Leftrightarrow-11-2\text{i}=z(\text{i}+2)\Leftrightarrow z=\dfrac{-11-2\text{i}}{\text{i}+2}\Leftrightarrow$

$\Leftrightarrow z=\dfrac{(-11-2\text{i})(2-\text{i})}{(2+\text{i})(2-\text{i})}\Leftrightarrow z=\dfrac{-22+11\text{i}-4\text{i}+2\text{i}^2}{4+1}\Leftrightarrow$

$\Leftrightarrow z=\dfrac{-24+7\text{i}}{5}\Leftrightarrow z=-\dfrac{24}{5}+\dfrac{7}{5}\text{i}$

A solução da equação é $z=-\dfrac{24}{5}+\dfrac{7}{5}\text{i}$.

24.2. Dado que z_2 é uma das raízes cúbicas de w:

$w=(z_2)^2=\left(4-\sqrt{3}\text{i}+\text{i}^{4n+2018}\right)^3=\left(4-\sqrt{3}\text{i}+\text{i}^{4n}\times\text{i}^{2018}\right)^3$

$$=\left(4-\sqrt{3}\,\mathrm{i}+\mathrm{i}^{2018}\right)^3=\left(4-\sqrt{3}\,\mathrm{i}+\mathrm{i}^{2}\right)^3=\left(4-\sqrt{3}\,\mathrm{i}-1\right)^3=$$

$$=\left(3-\sqrt{3}\,\mathrm{i}\right)^3=\left(3-\sqrt{3}\,\mathrm{i}\right)^2\left(3-\sqrt{3}\,\mathrm{i}\right)=$$

$$\begin{array}{r|l} 2018 & 4 \\ 2 & 504 \end{array}$$

$$=\left(9-6\sqrt{3}\,\mathrm{i}+3\mathrm{i}^2\right)\left(3-\sqrt{3}\,\mathrm{i}\right)=\left(6-6\sqrt{3}\,\mathrm{i}\right)\left(3-\sqrt{3}\,\mathrm{i}\right)=$$

$$=18-6\sqrt{3}\,\mathrm{i}+18\mathrm{i}^2=-24\sqrt{3}\,\mathrm{i}$$

25. O ponto D obtém-se do ponto A por uma reflexão de eixo Ox. Portanto, o número complexo cujo afixo é o ponto D é o conjugado de z, ou seja, $\overline{z}=\rho e^{-\mathrm{i}\theta}$.
O ponto B obtém-se do ponto D por uma reflexão central de centro O. Logo, o número complexo cujo afixo é o ponto B é o simétrico de $\overline{z}=\rho e^{-\mathrm{i}\theta}$, ou seja, é $-\overline{z}=\rho e^{-\mathrm{i}(\theta-\pi)}=\rho e^{\mathrm{i}(\pi-\theta)}$.
Finalmente, o ponto C obtém-se do ponto A por uma reflexão central de centro O. O número complexo cujo afixo é o ponto C é o simétrico de z, ou seja, $-z=\rho e^{\mathrm{i}(\pi+\theta)}$.

Pág. 123

26.1. $|z|\times z^2=z_1^3 \Leftrightarrow |z|\times\left(|z|e^{\mathrm{i}\theta}\right)^2=\left(\sqrt{3}-\mathrm{i}\right)^3 \Leftrightarrow$

$$\Leftrightarrow |z|\times|z|^2e^{\mathrm{i}2\theta}=\left(2e^{-\mathrm{i}\frac{\pi}{6}}\right)^3 \Leftrightarrow$$

$$\Leftrightarrow |z|^3e^{\mathrm{i}2\theta}=2^3e^{-\mathrm{i}\frac{3\pi}{6}} \Leftrightarrow$$

$$\Leftrightarrow |z|^3e^{\mathrm{i}2\theta}=8e^{-\mathrm{i}\frac{\pi}{2}} \Leftrightarrow$$

$$\Leftrightarrow |z|^3=8 \wedge 2\theta=-\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow z=2e^{-\mathrm{i}\frac{\pi}{4}} \vee z=2e^{\mathrm{i}\frac{3\pi}{4}}$$

$$z=\sqrt{3}-\mathrm{i}\ \text{e}\ \theta=\operatorname{Arg}(z)$$
$$|z|=\sqrt{3+1}=2$$
$$\begin{cases}\tan\theta=\dfrac{-1}{\sqrt{3}}=-\dfrac{\sqrt{3}}{3} \Rightarrow \theta=-\dfrac{\pi}{6}\\ \theta\in 4.^{\text{o}}\,Q\end{cases}$$
$$\sqrt{3}-\mathrm{i}=2e^{-\mathrm{i}\frac{\pi}{6}}$$

26.2. w é o inverso de $z_1\times z_2$.

$$w=\frac{1}{z_1\times z_2}=\frac{1}{\left(\sqrt{3}-\mathrm{i}\right)(-2\mathrm{i})}=$$

$$=\frac{1}{2e^{-\mathrm{i}\frac{\pi}{6}}\times 2e^{-\mathrm{i}\frac{\pi}{2}}}=\frac{1}{4e^{\mathrm{i}\left(-\frac{\pi}{6}-\frac{\pi}{2}\right)}}$$

$$=\frac{e^{\mathrm{i}\times 0}}{4e^{-\mathrm{i}\frac{2\pi}{3}}}=\frac{1}{4}e^{\mathrm{i}\left(0-\left(-\frac{2\pi}{3}\right)\right)}=\frac{1}{4}e^{\mathrm{i}\frac{2\pi}{3}}$$

$$z=\sqrt{3}-\mathrm{i}\ \text{e}\ \theta=\operatorname{Arg}(z)$$
$$|z|=\sqrt{3+1}=2$$
$$\begin{cases}\tan\theta=\dfrac{-1}{\sqrt{3}}=-\dfrac{\sqrt{3}}{3} \Rightarrow \theta=-\dfrac{\pi}{6}\\ \theta\in 4.^{\text{o}}\,Q\end{cases}$$
$$\sqrt{3}-\mathrm{i}=2e^{-\mathrm{i}\frac{\pi}{6}}$$
$$-2\mathrm{i}=2e^{-\mathrm{i}\frac{\pi}{2}}$$

27. $\dfrac{w}{z_1}=2+2\mathrm{i} \Leftrightarrow \dfrac{w}{1+\sqrt{3}\,\mathrm{i}}=2+2\mathrm{i} \Leftrightarrow w=(2+2\mathrm{i})\times\left(1+\sqrt{3}\,\mathrm{i}\right)$

$$\Leftrightarrow w=2\sqrt{2}\,e^{\mathrm{i}\frac{\pi}{4}}\times 2e^{\mathrm{i}\frac{\pi}{3}} \Leftrightarrow$$

$$\Leftrightarrow w=4\sqrt{2}\,e^{\mathrm{i}\left(\frac{\pi}{4}+\frac{\pi}{3}\right)} \Leftrightarrow w=4\sqrt{2}\,e^{\mathrm{i}\frac{7\pi}{12}}$$

$$|w|=4\sqrt{2}\ \text{e}\ \operatorname{Arg} w=\frac{7\pi}{12}$$

$$z_1=2+2\mathrm{i}=|z_1|e^{\mathrm{i}\theta_1}$$
$$|z_1|=\sqrt{4+4}=2\sqrt{2}$$
$$\theta_1=\arctan\frac{2}{2}=\frac{\pi}{4}$$
$$z_2=1+\sqrt{3}\,\mathrm{i}=|z_2|e^{\mathrm{i}\theta_2}$$
$$|z_2|=\sqrt{1+3}=2$$
$$\theta_2=\arctan\sqrt{3}=\frac{\pi}{3}$$

28. Seja $z=x+\mathrm{i}y$, então, $\overline{z}=x-\mathrm{i}y$. Substituindo, na equação:

$$\frac{2(x+\mathrm{i}y)+3\mathrm{i}(x-\mathrm{i}y+2)}{1+\mathrm{i}}=13+4\mathrm{i} \Leftrightarrow$$

$$\Leftrightarrow 2x+2\mathrm{i}y+3\mathrm{i}x+3y+6\mathrm{i}=13+13\mathrm{i}+4\mathrm{i}+4\mathrm{i}^2 \Leftrightarrow$$

$$\Leftrightarrow (2x+3y)+(3x+2y)\mathrm{i}+6\mathrm{i}=9+17\mathrm{i} \Leftrightarrow$$

$$\Leftrightarrow (2x+3y)+(3x+2y)\mathrm{i}=9+11\mathrm{i} \Leftrightarrow \begin{cases}2x+3y=9\\3x+2y=11\end{cases}$$

$$\Leftrightarrow \begin{cases}x=\dfrac{9-3y}{2}\\3\times\dfrac{9-3y}{2}+2y=11\end{cases} \Leftrightarrow \begin{cases}x=\dfrac{9-3y}{2}\\27-9y+4y=22\end{cases} \Leftrightarrow \begin{cases}x=3\\y=1\end{cases}$$

Logo, $z=3+\mathrm{i}$ é a solução da equação.

29. $z+2\mathrm{i}=\mathrm{i}z+k \Leftrightarrow z-\mathrm{i}z=k-2\mathrm{i} \Leftrightarrow z=\dfrac{k-2\mathrm{i}}{1-\mathrm{i}}$

$$\frac{w}{z}=2+2\mathrm{i} \Leftrightarrow w=z(2+2\mathrm{i}) \Leftrightarrow \qquad z=\frac{k-2\mathrm{i}}{1-\mathrm{i}}$$

$$\Leftrightarrow w=\left(\frac{k-2\mathrm{i}}{1-\mathrm{i}}\right)(2+2\mathrm{i}) \Leftrightarrow w=(k-2\mathrm{i})\left(\frac{2+2\mathrm{i}}{1-\mathrm{i}}\right) \Leftrightarrow$$

$$\Leftrightarrow w=(k-2\mathrm{i})\times\frac{(2+2\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})} \Leftrightarrow$$

$$\Leftrightarrow w=(k-2\mathrm{i})\times\frac{2+4\mathrm{i}-2}{2} \Leftrightarrow w=(k-2\mathrm{i})2\mathrm{i} \Leftrightarrow$$

$$\Leftrightarrow w=2k\mathrm{i}-4\mathrm{i}^2 \Leftrightarrow w=4+2k\mathrm{i}$$

Como $\operatorname{Im}(z)=8$, tem-se que $2k=8$, ou seja, $k=4$.

30. $|z+w|^2-|z-\overline{w}|^2=(z+w)\times\left(\overline{z+w}\right)-(z-\overline{w})\times\left(\overline{z-\overline{w}}\right)=$

$$=(z+w)\times(\overline{z}+\overline{w})-(z-\overline{w})\times(\overline{z}-w)=$$
$$=z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w}-z\overline{z}+zw+\overline{w}\,\overline{z}-\overline{w}w=$$
$$=(z\overline{w}+zw)+(w\overline{z}+\overline{w}\,\overline{z})=z(\overline{w}+w)+\overline{z}(w+\overline{w})=$$
$$=(z+\overline{z})(\overline{w}+w)=2\times\frac{z+\overline{z}}{2}\times 2\times\frac{\overline{w}+w}{2}$$
$$=4\times\operatorname{Re}(z)\times\operatorname{Re}(w)$$

$$\frac{z+\overline{z}}{2}=\operatorname{Re}(z)$$
$$\frac{\overline{w}+w}{2}=\operatorname{Re}(w)$$

31.1. $z=\dfrac{1+\mathrm{i}}{1-\mathrm{i}}=\dfrac{\sqrt{2}\,e^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{2}\,e^{-\mathrm{i}\frac{\pi}{4}}}=\dfrac{e^{\mathrm{i}\frac{\pi}{4}}}{e^{-\mathrm{i}\frac{\pi}{4}}}=e^{\mathrm{i}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right)}=e^{\mathrm{i}\frac{\pi}{2}}$

$$w=\frac{\sqrt{2}}{1-\mathrm{i}}=\frac{\sqrt{2}\,e^{\mathrm{i}\times 0}}{\sqrt{2}\,e^{-\mathrm{i}\frac{\pi}{4}}}=\frac{e^{\mathrm{i}\times 0}}{e^{-\mathrm{i}\frac{\pi}{4}}}=e^{\mathrm{i}\left(0-\left(-\frac{\pi}{4}\right)\right)}=e^{\mathrm{i}\frac{\pi}{4}}$$

$$z=1+\mathrm{i}\ \text{e}\ \theta=\operatorname{Arg}(z)$$
$$|z|=\sqrt{1+1}=\sqrt{2}$$
$$\begin{cases}\tan\theta=1\\ \theta\in 1.^{\text{o}}\,Q\end{cases} \Rightarrow \theta=\frac{\pi}{4}$$
$$z=\sqrt{2}\,e^{\mathrm{i}\frac{\pi}{4}}$$
$$1-\mathrm{i}=\overline{1+\mathrm{i}}=\sqrt{2}\,e^{-\mathrm{i}\frac{\pi}{4}}$$

31.2. O quadrilátero $[OABC]$ é um paralelogramo.

$$A\hat{O}C=\operatorname{Arg}(z)-\operatorname{Arg}(w)=\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\text{rad}=\frac{\pi}{4}\ \text{rad}$$

$$A\hat{O}B=B\hat{O}C=\frac{\frac{\pi}{4}}{2}\text{rad}=\frac{\pi}{8}\text{rad}$$

$$\operatorname{Arg}(z+w)=\left(\frac{\pi}{4}+\frac{\pi}{8}\right)\text{rad}=\frac{3\pi}{8}\text{rad}$$

$$z+w=\frac{1+\mathrm{i}}{1-\mathrm{i}}+\frac{\sqrt{2}}{1-\mathrm{i}}=\frac{\left(1+\sqrt{2}\right)+\mathrm{i}}{1-\mathrm{i}}=\frac{\left(\left(1+\sqrt{2}\right)+\mathrm{i}\right)(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})}$$

$$=\frac{1+\sqrt{2}+\left(1+\sqrt{2}\right)\mathrm{i}+\mathrm{i}+\mathrm{i}^2}{1-\mathrm{i}^2}=\frac{1+\sqrt{2}+\left(1+\sqrt{2}+1\right)\mathrm{i}-1^2}{1-\mathrm{i}^2}=$$

$$=\frac{\sqrt{2}+\left(2+\sqrt{2}\right)\mathrm{i}}{2}=\frac{\sqrt{2}}{2}+\frac{2+\sqrt{2}}{2}\mathrm{i}$$

$$\tan\left(\frac{3\pi}{8}\right)=\frac{\frac{2+\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=\frac{2+\sqrt{2}}{\sqrt{2}}=\frac{\left(2+\sqrt{2}\right)\sqrt{2}}{\left(\sqrt{2}\right)^2}=1+\sqrt{2}$$

Pág. 124

32.1. $\displaystyle\int\frac{x^2+3x-2}{\sqrt{x}}\mathrm{d}x=\int\left(\frac{x^2}{x^{\frac{1}{2}}}+\frac{3x}{x^{\frac{1}{2}}}-\frac{2}{x^{\frac{1}{2}}}\right)\mathrm{d}x=\int\left(x^{\frac{3}{2}}+3x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\right)\mathrm{d}x$

$$=\int x^{\frac{3}{2}}\mathrm{d}x+3\int x^{\frac{1}{2}}\mathrm{d}x-2\int x^{-\frac{1}{2}}\mathrm{d}x=\int x^{\frac{3}{2}}\mathrm{d}x+3\int x^{\frac{1}{2}}\mathrm{d}x-2\int x^{-\frac{1}{2}}\mathrm{d}x$$

$$=\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+3\times\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-2\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c,\ c\in\mathbb{R}=\frac{2}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}-4x^{\frac{1}{2}}+c=$$

$$=\frac{2}{5}x^2\sqrt{x}+2x\sqrt{x}-4\sqrt{x}+c,\ c\in\mathbb{R}$$

32.2. $\int\left(2e^x+\frac{6}{x}+\ln 2\right)dx=2\int e^x dx+6\int\frac{1}{x}dx+\ln 2\int dx=$

$=2e^x+6\ln|x|+(\ln 2)x+c\,,\ c\in\mathbb{R}$

33. $F(x)=\int\left(x^3-\frac{2}{x^2}+2\right)dx \Leftrightarrow F(x)=\int x^3dx-2\int x^{-2}dx+2\int dx$

$\Leftrightarrow F(x)=\frac{x^4}{4}-2\frac{x^{-1}}{-1}+2x+c\,,\ c\in\mathbb{R}\Leftrightarrow$

$\Leftrightarrow F(x)=\frac{1}{4}x^4+\frac{2}{x}+2x+c\,,\ c\in\mathbb{R}\Leftrightarrow$

$F(1)=3\Leftrightarrow\frac{1}{4}+2+2+c=3\Leftrightarrow c=3-4-\frac{1}{4}\Leftrightarrow c=-\frac{5}{4}$

$F(x)=\frac{1}{4}x^4+\frac{2}{x}+2x-\frac{5}{4}$

34.1. $\int(3-2x)e^{-x}dx=-e^{-x}(3-2x)-\int\left[-e^{-x}\times(-2)\right]dx=$

$=-e^{-x}(3-2x)-2\int e^{-x}dx=$

$u'=e^{-x}\,;\ u=-e^{-x}$
$v=3-2x;\ v'=-2$

$=-e^{-x}(3-2x)-2\left(-e^{-x}\right)+c\,,\ c\in\mathbb{R}=$

$=e^{-x}(-3+2x+2)+c\,,\ c\in\mathbb{R}==e^{-x}(2x-1)+c\,,\ c\in\mathbb{R}$

34.1. $\int(x+1)(x+2)^6dx=\frac{(x+2)^7}{7}\times(x+1)-\int\frac{(x+2)^7}{7}\times 1\,dx$

$u'=(x+2)^6$
$u=\frac{(x+2)^7}{7}$
$v=x+1;\ v'=1$

$=\frac{(x+2)^7}{7}\times(x+1)-\frac{1}{7}\times\frac{(x+2)^8}{8}+c\,,\ c\in\mathbb{R}=$

$=\frac{(x+2)^7}{7}\left(x+1-\frac{x+2}{8}\right)=\frac{(x+2)^7}{7}\times\frac{8x+8-x-2}{8}$

$=\frac{1}{56}(x+2)^7\times(7x+6)$

35. $\int_{\ln\frac{1}{2}}^{2}\left(e^t-\frac{1}{e^t}\right)dt=\int_{\ln\frac{1}{2}}^{2}e^t\,dt+\int_{\ln\frac{1}{2}}^{2}\left(-e^{-t}\right)dt=\left[e^t\right]_{\ln\frac{1}{2}}^{2}+\left[e^{-t}\right]_{\ln\frac{1}{2}}^{2}$

$=e^2-e^{\ln\frac{1}{2}}+e^{-2}-e^{-\ln\frac{1}{2}}=e^2-\frac{1}{2}+\frac{1}{e^2}-e^{\ln 2}=e^2+\frac{1}{e^2}-\frac{1}{2}-2$

$=e^2+\frac{1}{e^2}-\frac{5}{2}$

36. $f(x)=x^2+4$ e $g(x)=10-x$

$f(x)=g(x)\Leftrightarrow x^2+4=10-x\Leftrightarrow x^2+x-6=0\Leftrightarrow$

$\Leftrightarrow x=\frac{-1\pm\sqrt{1+24}}{2}\Leftrightarrow x=-3\vee x=2$

$g(x)=0\Leftrightarrow 10-x=0\Leftrightarrow x=10$

$A=\int_0^2 f(x)\,dx+\int_2^{10}f(x)\,dx=\int_0^2\left(x^2+4\right)dx+\int_2^{10}(10-x)\,dx$

$=\left[\frac{x^3}{3}+4x\right]_0^2+\left[10x-\frac{x^2}{2}\right]_2^{10}=$

$=\frac{2^3}{3}+4\times 2-0+\left(10\times 10-\frac{10^2}{2}\right)-\left(10\times 2-\frac{2^2}{2}\right)=$

$=\frac{8}{3}+8+100-50-20+2=\frac{8}{3}+40=\frac{128}{3}$

$A=\frac{128}{3}$ u.a.

36. $f(x)=x(x-2)=x^2-2x$ e $g(x)=4-x^2$

$f(x)=g(x)\Leftrightarrow x(x-2)=4-x^2\Leftrightarrow\sim$

$\Leftrightarrow x(x-2)-(2-x)(2+x)=0$

$\Leftrightarrow(x-2)(x+2+x)=0\Leftrightarrow x=2\vee x=-1$

$A=\int_{-1}^{2}\left[g(x)-f(x)\right]dx=\int_{-1}^{2}\left(4-x^2-x^2+2x\right)dx=$

$=\int_{-1}^{2}\left(-2x^2+2x+4\right)dx=\left[-2\frac{x^3}{3}+2\frac{x^2}{2}+4x\right]_{-1}^{2}=$

$=\left[-\frac{2}{3}x^3+x^2+4x\right]_{-1}^{2}=-\frac{2}{3}\times 8+4+8-\left(\frac{2}{3}+1-4\right)=$

$=-\frac{16}{3}-\frac{2}{3}+11=-6+15=9$

38.1. $f(x)=g(x)\Leftrightarrow x^2+4x-7=x^3-2x^2+5\Leftrightarrow$

$\Leftrightarrow x^3-3x^2-4x+12=0\Leftrightarrow\left(x^3-3x^2\right)-(4x-12)=0$

$\Leftrightarrow x^2(x-3)-4(x-3)=0$

38.2. $f(x)=g(x)\Leftrightarrow x^2(x-3)-4(x-3)=0\Leftrightarrow$

$\Leftrightarrow(x-3)\left(x^2-4\right)=0\Leftrightarrow x-3=0\vee x^2-4=0\Leftrightarrow$

$\Leftrightarrow x=3\vee x=-2\vee x=2$

$f(x)-g(x)=-x^3+3x^2+4x-12$

$A=\int_{-2}^{2}\left[g(x)-f(x)\right]dx+\int_2^3\left[f(x)-g(x)\right]dx$

$\int_{-2}^{2}\left[g(x)-f(x)\right]dx=\int_{-2}^{2}\left(x^3-3x^2-4x+12\right)dx=$

$=\left[\frac{x^4}{4}-3\frac{x^3}{3}-4\frac{x^2}{2}+12x\right]_{-2}^{2}=\left[\frac{x^4}{4}-x^3-2x^2+12x\right]_{-2}^{2}$

$=\left(\frac{16}{4}-8-8+24\right)-\left(\frac{16}{4}+8-8-24\right)=12+20=32$

$\int_2^3\left[f(x)-g(x)\right]dx=\int_2^3\left(-x^3+3x^2+4x-12\right)dx=$

$=\left[-\frac{x^4}{4}+x^3+2x^2-12x\right]_2^3=$

$=\left(-\frac{81}{4}+27+18-36\right)-\left(-\frac{16}{4}+8+8-24\right)=-\frac{81}{4}+21=\frac{3}{4}$

$A=\left(32+\frac{3}{4}\right)$ u.a. $=\frac{131}{4}$ u.a.

39. $\int_0^{\frac{\pi}{3}}\sin^3 x\,dx=\int_0^{\frac{\pi}{3}}\left(\sin x\times\sin^2 x\right)dx=\int_0^{\frac{\pi}{3}}\left[\sin x\left(1-\cos^2 x\right)\right]dx$

$=\int_0^{\frac{\pi}{3}}\left(\sin x-\sin x\cos^2 x\right)dx=\int_0^{\frac{\pi}{3}}\sin x\,dx+\int_0^{\frac{\pi}{3}}\left(-\sin x\cos^2 x\right)dx$

$=\left[-\cos x\right]_0^{\frac{\pi}{3}}+\left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{3}}=-\cos\frac{\pi}{3}+\cos 0+\frac{1}{3}\cos^3\frac{\pi}{3}-\frac{1}{3}\cos^3 0$

$=-\frac{1}{2}+1+\frac{1}{3}\times\frac{1}{8}-\frac{1}{3}\times 1=\frac{1}{2}+\frac{1}{24}-\frac{1}{3}=\frac{12+1-8}{24}=\frac{5}{24}$

40. $\int g(\theta)\,d\theta=\int(1+\cos\theta)^2\,d\theta=\int\left(1+2\cos\theta+\cos^2\theta\right)d\theta=$

$\cos(2\theta)=\cos^2\theta-\sin^2\theta$
$\cos(2\theta)=2\cos^2\theta-1$
$\cos^2\theta=\frac{1+\cos(2\theta)}{2}$

$=\int\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta=$

$=\int\left(1+2\cos\theta+\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta=$

$=\int\frac{3}{2}d\theta+2\int\cos\theta\,d\theta+\frac{1}{2}\times\frac{1}{2}\int 2\cos(2\theta)\,d\theta=$

$=\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin(2\theta)+c\,,\ c\in\mathbb{R}=$

$=\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\times 2\sin\theta\cos\theta+c\,,\ c\in\mathbb{R}=$

$=\frac{3}{2}\theta+2\sin\theta\left(1+\frac{\cos\theta}{4}\right)+c\,,\ c\in\mathbb{R}$

41. $a(t)=\cos(\pi t)$

$$v(t)=\int a(t)\,\mathrm{d}t=\int \cos(\pi t)\,\mathrm{d}t=$$

$$=\frac{1}{\pi}\int \pi\cos(\pi t)\,\mathrm{d}t=\frac{1}{\pi}\sin(\pi t)+c\ ,\ c\in\mathbb{R}$$

$$v(0)=\frac{1}{2\pi}\Leftrightarrow\frac{1}{\pi}\sin 0+c=\frac{1}{2\pi}\Leftrightarrow c=\frac{1}{2\pi}$$

$$v(t)=\frac{1}{\pi}\sin(\pi t)+\frac{1}{2\pi}$$

A função posição do ponto é dada por $s(t)=\int v(t)\,\mathrm{d}t$.

O deslocamento do ponto entre os instantes $t_1=0$ e $t_2=1{,}5$ é dado por $s(1{,}5)-s(0)$.

$$s(1{,}5)-s(0)=\int_0^{1{,}5} v(t)\,\mathrm{d}t=$$

$$=\int_0^{1{,}5}\left[\frac{1}{\pi}\sin(\pi t)+\frac{1}{2\pi}\right]\mathrm{d}t=$$

$$=-\frac{1}{\pi}\times\frac{1}{\pi}\int_0^{1{,}5}\left[-\pi\sin(\pi t)\right]\mathrm{d}t+\int_0^{1{,}5}\frac{1}{2\pi}\,\mathrm{d}t=$$

$$=-\frac{1}{\pi^2}\left[\cos(\pi t)\right]_0^{1{,}5}+\left[\frac{t}{2\pi}\right]_0^{1{,}5}=$$

$$-\frac{1}{\pi^2}\left[\cos(1{,}5\pi)-\cos 0\right]+\left[\frac{1{,}5}{2\pi}-0\right]=$$

$$-\frac{1}{\pi^2}(0-1)+\frac{1{,}5}{2\pi}=$$

$$=\frac{1}{\pi^2}+\frac{1{,}5}{2\pi}\approx 0{,}340$$

O deslocamento do ponto nos primeiros 1,5 s é aproximadamente igual a $0{,}340$ u.c.

42. $\dfrac{5x}{x^2+x-6}=\dfrac{A}{x-2}+\dfrac{B}{x+3}\Leftrightarrow$

$$\Leftrightarrow\frac{5x}{x^2+x-6}=\frac{A(x+3)+B(x-2)}{(x-2)(x+3)}\Leftrightarrow$$

$$\Leftrightarrow\frac{5x}{x^2+x-6}=\frac{Ax+3A+Bx-2B}{x^2+3x-2x-6}\Leftrightarrow$$

$$\Leftrightarrow\frac{5x}{x^2+x-6}=\frac{(A+B)x+3A-2B}{x^2+x-6}\Leftrightarrow$$

$$\Leftrightarrow(A+B)x+3A-2B=5x\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}A+B=5\\3A-2B=0\end{cases}\Leftrightarrow\begin{cases}B=5-A\\3A-10+2A=0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}B=5-A\\5A=10\end{cases}\Leftrightarrow\begin{cases}B=3\\A=2\end{cases}$$

Logo, $\dfrac{5x}{x^2+x-6}=\dfrac{2}{x-2}+\dfrac{3}{x+3}$

$$F(x)=\int\frac{5x}{x^2+x-6}\mathrm{d}x=\int\left(\frac{2}{x-2}+\frac{3}{x+3}\right)\mathrm{d}x=$$

$$=2\int\frac{1}{x-2}\mathrm{d}x+3\int\frac{1}{x+3}\mathrm{d}x=$$

$$=2\ln|x-2|+3\ln|x+3|+c\ ,\ c\in\mathbb{R}$$

$$F(3)=0\Leftrightarrow 2\ln|3-2|+3\ln|3+3|+c=0\Leftrightarrow$$

$$\Leftrightarrow 2\ln 1+3\ln 6+c=0\Leftrightarrow$$

$$\Leftrightarrow 2\times 0+3\ln 6+c=0\Leftrightarrow$$

$$\Leftrightarrow c=-3\ln 6$$

$$F(x)=2\ln|x-2|+3\ln|x+3|-3\ln 6$$