Ficha para praticar 11

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1.1. $2\sin\frac{\pi}{8}\cos\frac{\pi}{8}=\sin\left(2\times\frac{\pi}{8}\right)=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

1.2. $\cos^2\frac{5\pi}{8}-\sin^2\frac{5\pi}{8}=\cos\left(2\times\frac{5\pi}{8}\right)=$

$=\cos\frac{5\pi}{4}=\cos\left(\pi+\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$

1.3. $\cos\frac{5\pi}{8}\sin\frac{5\pi}{8}=\frac{2\sin\frac{5\pi}{8}\cos\frac{5\pi}{8}}{2}=\frac{\sin\left(2\times\frac{5\pi}{8}\right)}{2}=$

$=\frac{\sin\frac{5\pi}{4}}{2}=\frac{\sin\left(\pi+\frac{\pi}{4}\right)}{2}=\frac{-\sin\frac{\pi}{4}}{2}=\frac{-\frac{\sqrt{2}}{2}}{2}=-\frac{\sqrt{2}}{4}$

1.4. $\sin^2\frac{5\pi}{12}-\cos^2\frac{5\pi}{12}=-\left(\cos^2\frac{5\pi}{12}-\sin^2\frac{5\pi}{12}\right)=$

$=-\cos\left(2\times\frac{5\pi}{12}\right)=-\cos\frac{5\pi}{6}=$

$=-\cos\left(\pi-\frac{\pi}{6}\right)=-\left(-\cos\frac{\pi}{6}\right)=-\left(-\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{2}$

2. $\sin\alpha=\frac{4}{5}\wedge\alpha\in\left]\frac{\pi}{2},\frac{3\pi}{2}\right[\Rightarrow\alpha\in\left]\frac{\pi}{2},\pi\right[$

$\sin(2\alpha)=2\sin\alpha\cos\alpha$; $\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha$ e

$\tan(2\alpha)=\frac{\sin(2\alpha)}{\cos(2\alpha)}$.

Pela fórmula fundamental da trigonometria, tem-se:
$\sin^2\alpha+\cos^2\alpha=1$, ou seja,

$\left(\frac{4}{5}\right)^2+\cos^2\alpha=1\Leftrightarrow\cos^2\alpha=1-\frac{16}{25}\Leftrightarrow$

$\Leftrightarrow\cos^2\alpha=\frac{9}{25}\Leftrightarrow\cos\alpha=-\frac{3}{5}\vee\cos\alpha=\frac{3}{5}$, como

$\alpha\in\left]\frac{\pi}{2},\pi\right[$, $\cos\alpha<0$, pelo que $\cos\alpha=-\frac{3}{5}$.

$\sin(2\alpha)=2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)=-\frac{24}{25}$

$\cos(2\alpha)=\frac{9}{25}-\frac{16}{25}=-\frac{7}{25}$; $\tan(2\alpha)=\frac{-\frac{24}{25}}{-\frac{7}{25}}=\frac{24}{7}$

$\sin(2\alpha)-\cos(2\alpha)+\tan(2\alpha)=-\frac{24}{25}-\left(-\frac{7}{25}\right)+\frac{24}{7}=\frac{481}{175}$

3. $\cos(2\theta)=\cos^2\theta-\sin^2\theta\Leftrightarrow\cos(2\theta)=1-\sin^2\theta-\sin^2\theta$

$\Leftrightarrow\cos(2\theta)=1-2\sin^2\theta\Leftrightarrow$. Como $\cos(2\theta)=\frac{1}{3}$:

$\frac{1}{3}=1-2\sin\theta\Leftrightarrow2\sin^2\theta=1-\frac{1}{3}\Leftrightarrow$

$\Leftrightarrow2\sin^2\theta=\frac{2}{3}\Leftrightarrow\sin^2\theta=\frac{1}{3}\Leftrightarrow\sin\theta=-\sqrt{\frac{1}{3}}\vee\sin\theta=\sqrt{\frac{1}{3}}\Leftrightarrow$

$\Leftrightarrow\sin\theta=-\frac{\sqrt{3}}{3}\vee\sin\theta=\frac{\sqrt{3}}{3}$

$0<2\theta<\pi$, isto é, $0<\theta<\frac{\pi}{2}$ pelo que $\sin\theta=\frac{\sqrt{3}}{3}$.

4. $\sin(a+b)=\sin a\cos b+\sin b\cos a$ **(1)**

Determinemos $\cos a$ e $\sin b$.
Pela fórmula fundamental da trigonometria:

- $\cos^2 a+\sin^2 a=1$, isto é, $\cos^2 a+\left(\frac{2\sqrt{2}}{3}\right)^2=1\Leftrightarrow$

  $\Leftrightarrow\cos^2 a=1-\frac{8}{9}\Leftrightarrow$

  $\Leftrightarrow\cos^2 a=\frac{1}{9}\Leftrightarrow\cos a=-\frac{1}{3}\vee\cos a=\frac{1}{3}$

  Como $a\in\left]\frac{\pi}{2},\pi\right[$, $\cos a<0$ pelo que $\cos a=-\frac{1}{3}$.

- $\cos^2 b+\sin^2 b=1$, isto é, $\left(\frac{3}{5}\right)^2+\sin^2 b=1\Leftrightarrow$

  $\Leftrightarrow\sin^2 b=1-\frac{9}{25}\Leftrightarrow\sin^2 b=\frac{16}{25}\Leftrightarrow$

  $\Leftrightarrow\sin b=-\frac{4}{5}\vee\sin b=\frac{4}{5}$, como $b\in\left]-\frac{\pi}{2},0\right[$,

  $\sin b<0$ pelo que $\sin b=-\frac{4}{5}$.

Voltando a **(1)**:

$\sin(a+b)=\frac{2\sqrt{2}}{3}\times\frac{3}{5}+\left(-\frac{4}{5}\right)\left(-\frac{1}{3}\right)=\frac{2\sqrt{2}}{5}+\frac{4}{15}$

5. $\cos\left(\frac{\pi}{4}-\beta\right)=\cos\frac{\pi}{4}\cos\beta+\sin\frac{\pi}{4}\sin\beta$ **(1)**

$\sin^2\beta+\left(-\frac{4}{5}\right)^2=1\Leftrightarrow\sin^2\beta=1-\frac{16}{25}\Leftrightarrow$

$\Leftrightarrow\sin^2\beta=\frac{9}{25}\Leftrightarrow\sin\beta=-\frac{3}{5}\vee\sin\beta=\frac{3}{5}$

Como $\beta\in\left]\pi,\frac{3\pi}{2}\right[$, $\sin\beta<0$, pelo que $\sin\beta=-\frac{3}{5}$.

Voltando a **(1)**:

$\cos\left(\frac{\pi}{4}-\beta\right)=\frac{\sqrt{2}}{2}\left(-\frac{4}{5}\right)+\frac{\sqrt{2}}{2}\left(-\frac{3}{5}\right)=$

$=-\frac{4\sqrt{2}}{10}-\frac{3\sqrt{2}}{10}=-\frac{7\sqrt{2}}{10}$

6. $\sin\left(\frac{\pi}{4}-\alpha\right)=\sin\frac{\pi}{4}\cos\alpha-\sin\alpha\cos\frac{\pi}{4}=$

$=\frac{\sqrt{2}}{2}\cos\alpha-\sin\alpha\frac{\sqrt{2}}{2}=$

$=\frac{\sqrt{2}}{2}(\cos\alpha-\sin\alpha)$

Como $\cos\alpha-\sin\alpha=\frac{1}{5}$, substituindo:

$\sin\left(\frac{\pi}{4}-\alpha\right)=\frac{\sqrt{2}}{2}\times\frac{1}{5}=\frac{\sqrt{2}}{10}$

7. $\cos\frac{\pi}{12}=\cos\left(\frac{4\pi}{12}-\frac{3\pi}{12}\right)=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=$

$=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}=$

$=\frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}=\frac{\sqrt{2}+\sqrt{6}}{4}$

$$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\cos\frac{\pi}{3}=$$
$$=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=$$
$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$
$$\cos\frac{\pi}{12}+\sin\frac{\pi}{12}=\frac{\sqrt{2}+\sqrt{6}}{4}+\frac{\sqrt{6}-\sqrt{2}}{4}=$$
$$=\frac{\sqrt{2}+\sqrt{6}+\sqrt{6}-\sqrt{2}}{4}=\frac{2\sqrt{6}}{4}=\frac{\sqrt{6}}{2}$$

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8.1. $4\cos^2\alpha-\sin^2(2\alpha)=4\cos^4(\alpha)=$
$$=4\cos^2\alpha-\left[\sin(2\alpha)\right]^2=$$
$$=4\cos^2\alpha-\left(2\sin(\alpha)\cos(\alpha)\right)^2=$$
$$=4\cos^2\alpha-4\sin^2\alpha\cos^2\alpha=$$
$$=4\cos^2\alpha\left(1-\sin^2(\alpha)\right)=$$
$$=4\cos^2\alpha\left(\cos^2(\alpha)\right)=4\cos^4\alpha$$

8.2. $\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}=\dfrac{1-\dfrac{\sin^2\alpha}{\cos^2\alpha}}{1+\dfrac{\sin^2\alpha}{\cos^2\alpha}}=\dfrac{\dfrac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha}}{\dfrac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}}=$
$$=\frac{\cos^2\alpha-\sin^2\alpha}{1}=\cos(2\alpha)$$

8.3. $\cos^4(3\alpha)-\sin^4(3\alpha)=$
$$=\left[\cos^2(3\alpha)\right]^2-\left[\sin^2(3\alpha)\right]^2=$$
$$=\left[\cos^2(3\alpha)-\sin^2(3\alpha)\right]\times\left[\cos^2(3\alpha)+\sin^2(3\alpha)\right]$$
$$=\left[\cos^2(3\alpha)-\sin^2(3\alpha)\right]\times 1$$
$$=\cos^2(3\alpha)-\sin^2(3\alpha)=$$
$$=\cos(2\times 3\alpha)\quad=\cos(6\alpha)$$

8.4. $\dfrac{2\tan\alpha}{1+\tan^2\alpha}=\dfrac{2\dfrac{\sin\alpha}{\cos\alpha}}{\dfrac{1}{\cos^2\alpha}}=\dfrac{2\sin\alpha\cos^2\alpha}{\cos\alpha}=$
$$=2\sin\alpha\cos\alpha=\sin(2\alpha)$$

9.1. $\sin x\cos x=\dfrac{1}{4}\Leftrightarrow 2\sin x\cos x=2\times\dfrac{1}{4}\Leftrightarrow$
$$\Leftrightarrow\sin(2x)=\frac{1}{2}\Leftrightarrow\sin(2x)=\sin\frac{\pi}{6}$$
$$\Leftrightarrow 2x=\frac{\pi}{6}+2k\pi\vee 2x=\pi-\frac{\pi}{6}+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$
$$\Leftrightarrow x=\frac{\pi}{12}+k\pi\vee x=\frac{5\pi}{12}+k\pi,\ k\in\mathbb{Z}$$

9.2. $\cos^2 x-\sin^2 x=-\dfrac{\sqrt{3}}{2}\Leftrightarrow\cos(2x)=-\dfrac{\sqrt{3}}{2}\Leftrightarrow$
$$\Leftrightarrow\cos(2x)=\cos\left(\frac{5\pi}{6}\right)\Leftrightarrow$$
$$\Leftrightarrow 2x=\frac{5\pi}{6}+k2\pi\vee 2x=-\frac{5\pi}{6}+k2\pi,\ k\in\mathbb{Z}\Leftrightarrow$$
$$\Leftrightarrow x=\frac{5\pi}{12}+k\pi\vee x=-\frac{5\pi}{12}+k\pi,\ k\in\mathbb{Z}$$

9.3. $\dfrac{1}{2}\cos x-\dfrac{\sqrt{3}}{2}\sin x=\dfrac{1}{2}\Leftrightarrow\cos\dfrac{\pi}{3}\cos x-\sin\dfrac{\pi}{3}\sin x=\dfrac{1}{2}\Leftrightarrow$
$$\Leftrightarrow\cos\left(\frac{\pi}{3}+x\right)=\frac{1}{2}\Leftrightarrow\cos\left(\frac{\pi}{3}+x\right)=\cos\frac{\pi}{3}\Leftrightarrow$$
$$\Leftrightarrow\frac{\pi}{3}+x=\frac{\pi}{3}+2k\pi\vee\frac{\pi}{3}+x=-\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$
$$\Leftrightarrow x=2k\pi\vee x=-\frac{2\pi}{3}+2k\pi,\ \ k\in\mathbb{Z}$$

9.4. $\sin x=-\cos x-1\Leftrightarrow\sin x+\cos x=-1\Leftrightarrow$
$$\Leftrightarrow\frac{\sqrt{2}}{2}\sin x+\frac{\sqrt{2}}{2}\cos x=-\frac{\sqrt{2}}{2}\Leftrightarrow$$
$$\Leftrightarrow\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x=-\cos\frac{\sqrt{2}}{2}\Leftrightarrow$$
$$\Leftrightarrow\sin\left(x+\frac{\pi}{4}\right)=\sin\left(-\frac{\pi}{4}\right)\Leftrightarrow$$
$$\Leftrightarrow x+\frac{\pi}{4}=-\frac{\pi}{4}+\vee x+\frac{\pi}{4}=\pi-\left(-\frac{\pi}{4}\right)+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$
$$\Leftrightarrow x=-\frac{\pi}{2}+2k\pi\vee x=\pi+2k\pi,\ k\in\mathbb{Z}$$

9.5. $\sin x=-\sin\dfrac{x}{2}\Leftrightarrow\sin x+\sin\dfrac{x}{2}=0\Leftrightarrow$
$$\Leftrightarrow 2\sin\frac{x}{2}\cos\frac{x}{2}+\sin\frac{x}{2}=0\Leftrightarrow\sin\frac{x}{2}\left(2\cos\frac{x}{2}+1\right)=0\Leftrightarrow$$
$$\Leftrightarrow\sin\frac{x}{2}=0\vee 2\cos\frac{x}{2}+1=0\Leftrightarrow$$
$$\Leftrightarrow\sin\frac{x}{2}=0\vee\cos\frac{x}{2}=-\frac{1}{2}$$
$$\Leftrightarrow\frac{x}{2}=k\pi\vee\frac{x}{2}=\frac{2\pi}{3}+2k\pi\vee\frac{x}{2}=-\frac{2\pi}{3}+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$
$$\Leftrightarrow x=2k\pi\vee x=\frac{4\pi}{3}+4k\pi\vee x=-\frac{4\pi}{3}+4k\pi,\ \mathbb{Z}$$

9.6. $\cos(2x)-\sin x=0\Leftrightarrow\cos^2 x-\sin^2 x-\sin x=0\Leftrightarrow$
$$\Leftrightarrow 1-\sin^2 x-\sin^2 x-\sin x=0\Leftrightarrow$$
$$\Leftrightarrow -2\sin^2 x-\sin x+1=0\Leftrightarrow$$
$$\Leftrightarrow 2\sin^2 x+\sin x-1=0\Leftrightarrow$$
$$\Leftrightarrow\sin x=\frac{-1\pm\sqrt{1-4\times 2\times(-1)}}{2\times 2}\Leftrightarrow$$
$$\Leftrightarrow\sin x=\frac{-1+3}{4}\vee\sin x=\frac{-1-3}{4}\Leftrightarrow$$
$$\Leftrightarrow\sin x=\frac{1}{4}\vee\sin x=-1\Leftrightarrow$$
$$\Leftrightarrow\sin x=\sin\frac{\pi}{6}\vee\sin x=\sin\left(-\frac{\pi}{2}\right)\Leftrightarrow$$
$$\Leftrightarrow x=\frac{\pi}{6}+2k\pi\vee x=\frac{5\pi}{6}+2k\pi\vee x=-\frac{\pi}{2}+2k\pi,k\in\mathbb{Z}$$

9.7. $\cos(2x)-5\cos x+3=0\Leftrightarrow\cos^2 x-\sin^2 x-5\cos x+3=0\Leftrightarrow$
$$\Leftrightarrow\cos^2 x-\left(1-\cos^2 x\right)-5\cos x+3=0\Leftrightarrow$$
$$\Leftrightarrow\cos^2 x-1+\cos^2 x-5\cos x+3=0\Leftrightarrow$$
$$\Leftrightarrow 2\cos^2 x-5\cos x+2=0\Leftrightarrow$$
$$\Leftrightarrow\cos x=\frac{5\pm\sqrt{25-4\times 2\times 2}}{2\times 2}\Leftrightarrow$$
$$\Leftrightarrow\cos x=2\vee\cos x=\frac{1}{2}\Leftrightarrow$$

$\Leftrightarrow \cos x = \frac{1}{2} \Leftrightarrow \cos x = \cos\frac{\pi}{3} \Leftrightarrow$

$\Leftrightarrow x = \frac{\pi}{3} + 2k\pi \vee x = -\frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$

9.8. $\cos x + \sqrt{3}\sin x = 1 \Leftrightarrow \frac{1}{2}\cos + \frac{\sqrt{3}}{2}\sin x = \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x = \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \sin\left(\frac{\pi}{6} + x\right) = \sin\frac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow \frac{\pi}{6} + x = \frac{\pi}{6} + 2k\pi \vee \frac{\pi}{6} + x = \pi - \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x = 2k\pi \vee x = \frac{2\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$

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10.1. $\sqrt{3}\cos x + \sin x = 1 \Leftrightarrow \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \sin\frac{\pi}{3}\cos x + \cos\frac{\pi}{3}\sin x = \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \sin\left(\frac{\pi}{3} + x\right) = \sin\frac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow \frac{\pi}{3} + x = \frac{\pi}{6} + 2k\pi \vee \frac{\pi}{3} + x = \pi - \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x = -\frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z} \vee x = \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$

Como $x \in \left]-\pi, \frac{\pi}{2}\right]$:

$x = -\frac{\pi}{6} + 2k\pi$	$x = \frac{\pi}{2} + 2k\pi$
$k = 0 \Rightarrow x = -\frac{\pi}{6}$ ⬅	$k = 0 \Rightarrow x = \frac{\pi}{2}$ ⬅
$k = 1 \Rightarrow x = \frac{11\pi}{6}$	$k = 1 \Rightarrow x = \frac{5\pi}{2}$
$k = -1 \Rightarrow x = -\frac{13\pi}{6}$	$k = -1 \Rightarrow x = -\frac{3\pi}{2}$

Portanto, $x = -\frac{\pi}{6} \vee x = \frac{\pi}{2}$

10.2. $1 = \cos x + \sin\frac{x}{2} \Leftrightarrow 1 = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} + \sin\frac{x}{2} \Leftrightarrow$

$\Leftrightarrow 1 = 1 - \sin^2\frac{x}{2} - \sin^2\frac{x}{2} + \sin\frac{x}{2} \Leftrightarrow$

$\Leftrightarrow 0 = -2\sin^2\frac{x}{2} + \sin\frac{x}{2} \Leftrightarrow$

$\Leftrightarrow \sin\frac{x}{2}\left(-2\sin\frac{x}{2} + 1\right) = 0 \Leftrightarrow$

$\Leftrightarrow \sin\frac{x}{2} = 0 \vee -2\sin\frac{x}{2} + 1 = 0 \Leftrightarrow$

$\Leftrightarrow \sin\frac{x}{2} = 0 \vee \sin\frac{x}{2} = \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \sin\frac{x}{2} = 0 \vee \sin\frac{x}{2} = \sin\frac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow \frac{x}{2} = k\pi \vee \frac{x}{2} = \frac{\pi}{6} + 2k\pi \vee \frac{x}{2} = \pi - \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x = 2k\pi \vee x = \frac{\pi}{3} + 4k\pi \vee x = \frac{5\pi}{3} + 4k\pi,\ k \in \mathbb{Z}$

Como $x \in \left]-\frac{\pi}{2}, \frac{5\pi}{3}\right[$:

$x = 2k\pi$	$x = \frac{\pi}{3} + 4k\pi$ ⬅
$k = 0 \Rightarrow x = 0$ ⬅	$k = 0 \Rightarrow x = \frac{\pi}{3}$
$k = 1 \Rightarrow x = 2\pi$	$k = 1 \Rightarrow x = \frac{\pi}{3} + 4\pi$
$k = -1 \Rightarrow x = -2\pi$	$k = -1 \Rightarrow x = \frac{\pi}{3} - 4\pi$

$x = \frac{5\pi}{3} + 4k\pi$ ⬅

$k = 0 \Rightarrow x = \frac{5\pi}{3}$; $k = -1 \Rightarrow x = \frac{5\pi}{3} - 4\pi$

Portanto, $x = 0 \vee x = \frac{\pi}{3}$.

10.3. $\sin\frac{\pi}{8}\cos x + \cos\frac{\pi}{8}\sin x = 1 \Leftrightarrow \sin\left(\frac{\pi}{8} + x\right) = 1 \Leftrightarrow$

$\Leftrightarrow \frac{\pi}{8} + x = \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x = \frac{\pi}{2} - \frac{\pi}{8} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow x = \frac{3\pi}{8} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

Como $x \in]0, \pi[$, vem $x = \frac{3\pi}{8}$. Logo, $x = \frac{3\pi}{8}$.

10.4. $\cos\left(2x + \frac{\pi}{4}\right) = \cos^2(3x) - \sin^2(3x) \Leftrightarrow$

$\Leftrightarrow \cos\left(2x + \frac{\pi}{4}\right) = \cos(2 \times 3x) \Leftrightarrow$

$\Leftrightarrow \cos\left(2x + \frac{\pi}{4}\right) = \cos(6x) \Leftrightarrow$

$\Leftrightarrow 2x + \frac{\pi}{4} = 6x + 2k\pi \vee 2x + \frac{\pi}{4} = -6x + k2\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow -4x = -\frac{\pi}{4} + 2k\pi \vee 8x = -\frac{\pi}{4} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x = \frac{\pi}{16} - \frac{k\pi}{2} \vee x = -\frac{\pi}{32} + \frac{k\pi}{4},\ k \in \mathbb{Z}$

$x = \frac{\pi}{16} - \frac{k\pi}{2} = \frac{\pi - 8k\pi}{16}$	$x = -\frac{\pi}{32} + \frac{k\pi}{4} = \frac{-\pi + 8k\pi}{32}$
$k = 0 \Rightarrow x = \frac{\pi}{16}$ ⬅	$k = 0 \Rightarrow x = -\frac{\pi}{32}$ ⬅
$k = 1 \Rightarrow x = -\frac{7\pi}{16}$ ⬅	$k = 1 \Rightarrow x = \frac{7\pi}{32}$ ⬅
$k = 2 \Rightarrow x = -\frac{15\pi}{16}$	$k = 2 \Rightarrow x = \frac{15\pi}{32}$ ⬅
$k = -1 \Rightarrow x = \frac{9\pi}{16}$ ⬅	$k = 3 \Rightarrow x = \frac{23\pi}{32}$ ⬅
$k = -2 \Rightarrow x = \frac{17\pi}{16}$	$k = 4 \Rightarrow x = \frac{31\pi}{32}$ ⬅
	$k = 5 \Rightarrow x = -\frac{39\pi}{32}$
	$k = -1 \Rightarrow x = -\frac{9\pi}{32}$
	$k = -2 \Rightarrow x = -\frac{17\pi}{32}$

Portanto:

$$x=-\frac{7\pi}{16}\vee x=-\frac{9\pi}{32}\vee x=-\frac{\pi}{32}\vee x=\frac{\pi}{16}\vee x=\frac{7\pi}{32}\vee$$

$$\vee x=\frac{15\pi}{32}\vee x=\frac{9\pi}{16}\vee x=\frac{23\pi}{32}\vee x=\frac{31\pi}{32}$$

10.5. $\sqrt{2}\left(\cos x+\sin x\right)=1\Leftrightarrow \sqrt{2}\cos x+\sqrt{2}\sin x=1\Leftrightarrow$

$$\Leftrightarrow \frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\sin x=\frac{1}{2}\Leftrightarrow$$

$$\Leftrightarrow \cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x=\frac{1}{2}\Leftrightarrow$$

$$\Leftrightarrow \cos\left(\frac{\pi}{4}-x\right)=\cos\frac{\pi}{3}\Leftrightarrow$$

$$\Leftrightarrow \frac{\pi}{4}-x=\frac{\pi}{3}+2k\pi\vee\frac{\pi}{4}-x=-\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$

$$\Leftrightarrow -x=\frac{\pi}{3}-\frac{\pi}{4}+2k\pi\vee -x=-\frac{\pi}{3}-\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$

$$\Leftrightarrow x=-\frac{\pi}{12}-2k\pi\vee x=\frac{7\pi}{12}-2k\pi,\ k\in\mathbb{Z}$$

Como $x\in\left]-\frac{\pi}{2},\frac{\pi}{6}\right[$, vem que

$x=-\frac{\pi}{12}+2k\pi$	$x=\frac{7\pi}{12}+2k\pi$
$k=0\Rightarrow x=-\frac{\pi}{12}$ ⬅	$k=0\Rightarrow x=\frac{7\pi}{12}$
$k=1\Rightarrow x=\frac{23\pi}{12}$	$k=-1\Rightarrow x=-\frac{17\pi}{12}$
$k=-1\Rightarrow x=-\frac{25\pi}{12}$	

Portanto, $x=-\frac{\pi}{12}$

11. $x\in\left]\pi,2\pi\right[\Leftrightarrow\frac{x}{2}\in\left]\frac{\pi}{2},\pi\right[$

$$3\cos\frac{x}{2}=-\sqrt{2}\Leftrightarrow\cos\frac{x}{2}=-\frac{\sqrt{2}}{3}$$

$$\cos^2\frac{x}{2}+\sin^2\frac{x}{2}=1\quad\left(-\frac{\sqrt{2}}{3}\right)^2+\sin^2\frac{x}{2}=1\Leftrightarrow\sin^2\frac{x}{2}=1-\frac{2}{9}\Leftrightarrow$$

$$\Leftrightarrow\sin^2\frac{x}{2}=\frac{7}{9}\Leftrightarrow\sin\frac{x}{2}=\frac{\sqrt{7}}{3}\vee\sin\frac{x}{2}=-\frac{\sqrt{7}}{3}$$

Se $\cos\left(\frac{x}{2}\right)=-\frac{\sqrt{2}}{2}<0\wedge x\in\left]\frac{\pi}{2},\pi\right[$

então, $\sin\frac{x}{2}>0$, pelo que, $\sin\frac{x}{2}=\frac{\sqrt{7}}{3}$.

$$\cos\left(x-\pi\right)+\cos\left(-\frac{\pi}{2}+x\right)=-\cos x+\sin x=$$

$$=-\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)+2\sin\frac{x}{2}\cos\frac{x}{2}=$$

$$=-\left(\frac{2}{9}-\frac{7}{9}\right)+2\left(\frac{\sqrt{7}}{3}\right)\left(-\frac{\sqrt{2}}{3}\right)=$$

$$=\frac{5}{9}-\frac{2\sqrt{14}}{9}=\frac{5-2\sqrt{14}}{9}$$

12.1. Seja M o ponto médio de $[AC]$, tem-se que

$$\tan\left(2x\right)=\frac{\overline{BM}}{2}\Leftrightarrow BM=2\tan\left(2x\right)$$

$$\text{Área }\Delta[ABC]=\frac{\overline{AC}\times\overline{BM}}{2}=\frac{4\times2\tan\left(2x\right)}{2}=4\tan\left(2x\right)$$

Portanto, $A\left(x\right)=4\tan\left(2x\right),\ x\in\left]0,\frac{\pi}{4}\right[$.

12.2. $\sin\left(\frac{\pi}{2}+\beta\right)=\frac{4}{5}\Leftrightarrow\cos\beta=\frac{4}{5}$

$$\sin^2\beta+\left(\frac{4}{5}\right)^2=1\Leftrightarrow\sin^2\beta=1-\frac{16}{25}\Leftrightarrow$$

$$\Leftrightarrow\sin^2\beta=\frac{9}{25}\Rightarrow\sin\beta=\frac{3}{5}\text{, pois }\beta\in1.^{\text{o}}\text{Q}$$

$$\sin\left(2\beta\right)=2\sin\beta\cos\beta=2\times\frac{3}{5}\times\frac{4}{5}=\frac{24}{25}$$

$$\cos\left(2\beta\right)=\cos^2\beta-\sin^2\beta=$$

$$=\left(\frac{4}{5}\right)^2-\left(\frac{3}{5}\right)^2=\frac{16}{25}-\frac{9}{25}=\frac{7}{25}$$

$$\tan\left(2\beta\right)=\frac{\sin\left(2\beta\right)}{\cos\left(2\beta\right)}=\frac{\frac{24}{25}}{\frac{7}{25}}=\frac{24\times25}{7\times25}=\frac{24}{7}$$

$$A\left(\beta\right)=4\tan\left(2\beta\right)=4\times\frac{24}{7}=\frac{96}{7}$$

Portanto, $A\left(\beta\right)=\frac{96}{7}$.

13.1. $\sin x\cos\frac{\pi}{3}-\cos x\sin\frac{\pi}{3}>0\wedge x\in\left]0,2\pi\right[\Leftrightarrow$

$$\Leftrightarrow\sin\left(x-\frac{\pi}{3}\right)>0\wedge x-\frac{\pi}{3}\in\left]-\frac{\pi}{3},\frac{5\pi}{3}\right[\Leftrightarrow$$

$$\Leftrightarrow0<x-\frac{\pi}{3}<\pi\Leftrightarrow\frac{\pi}{3}<x<\pi+\frac{\pi}{3}\Leftrightarrow$$

$$\Leftrightarrow x\in\left]\frac{\pi}{3},\frac{4\pi}{3}\right[$$

13.2. $\cos\left(2x\right)+3\cos x\geq1\wedge x\in\left]-\pi,\pi\right[$

$$\Leftrightarrow\cos^2x-\sin^2x+3\cos x\geq1\wedge x\in\left]-\pi,\pi\right[\Leftrightarrow$$

$$\Leftrightarrow\cos^2x-\left(1-\cos^2x\right)+3\cos x\geq1\wedge x\in\left]-\pi,\pi\right[\Leftrightarrow$$

$$\Leftrightarrow2\cos^2x+3\cos x-2\geq0\wedge x\in\left]-\pi,\pi\right[\Leftrightarrow$$

Cálculo auxiliar:

$$2\cos^2x+3\cos x-2=0$$

$$\Leftrightarrow\cos x=\frac{-3\pm\sqrt{9-4\times2\times(-2)}}{2\times2}$$

$$\Leftrightarrow\cos x=\frac{-3+5}{4}\vee\cos x=\frac{-3-5}{4}$$

$$\Leftrightarrow\cos x=\frac{1}{2}\vee\cos x=-2$$

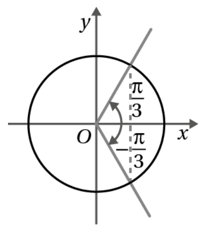
$$2\cos^2x+3\cos x-2\geq0\wedge x\in\left]-\pi,\pi\right[\Leftrightarrow$$

$$\Leftrightarrow\left(\cos x\leq-2\vee\cos x\geq\frac{1}{2}\right)\wedge x\in\left]-\pi,\pi\right[\Leftrightarrow$$

$$\Leftrightarrow\left(x\in\varnothing\vee\cos x\geq\frac{1}{2}\right)\wedge x\in\left]-\pi,\pi\right[\Leftrightarrow$$

$$\Leftrightarrow\cos x\geq\frac{1}{2}\wedge x\in\left]-\pi,\pi\right[\Leftrightarrow$$

$$\Leftrightarrow-\frac{\pi}{3}\leq x\leq\frac{\pi}{3}$$



Portanto, $x\in\left]-\frac{\pi}{3},\frac{\pi}{3}\right[$.

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14. Área $\Delta[PQR]=\dfrac{\overline{RQ}\times\overline{PQ}}{2}$

$\cos\alpha=\overline{OQ}$; $\sin\alpha=\overline{PQ}$

$\overline{RQ}=\overline{RO}+\overline{OQ}=1+\cos\alpha$

Assim:

$$A(x)=\frac{(1+\cos x)\sin x}{2}=\frac{\sin x+\sin x\cos x}{2}=$$
$$=\frac{2(\sin x+\sin x\cos x)}{2\times 2}=$$
$$=\frac{2\sin x+2\sin x\cos x}{4}=\frac{2\sin x+\sin(2x)}{4}$$

Portanto, $A(x)=\dfrac{2\sin x+\sin(2x)}{4}$.

15.1. Perímetro $\Delta[ABC]=\overline{AB}+\overline{BC}+\overline{AC}$

$$\tan x=\frac{\overline{BC}}{\overline{AB}}\Leftrightarrow\tan x=\frac{\overline{BC}}{1}\Leftrightarrow\tan x=\overline{BC}$$

$$\cos x=\frac{\overline{AB}}{\overline{AC}}\Leftrightarrow\cos x=\frac{1}{\overline{AC}}\Leftrightarrow\overline{AC}=\frac{1}{\cos x}$$

$$f(x)=1+\tan x+\frac{1}{\cos x}=$$
$$=1+\frac{\sin x}{\cos x}+\frac{1}{\cos x}=\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}+\frac{1}{\cos x}=$$
$$=\frac{1+\sin x+\cos x}{\cos x}$$

Portanto, $f(x)=\dfrac{1+\sin x+\cos x}{\cos x},\ x\in\left]0,\dfrac{\pi}{2}\right[$.

15.2. $f(2x)=\dfrac{1+\sin(2x)+\cos(2x)}{\cos(2x)}=$

$$=\frac{1+2\sin x\cos+\cos^2x-\sin^2x}{\cos^2x-\sin^2x}=$$
$$=\frac{1-\sin^2x+\cos^2x+2\sin x\cos x}{(\cos x-\sin x)(\cos x+\sin x)}=$$
$$=\frac{\cos^2x+\cos^2x+2\sin x\cos x}{(\cos x-\sin x)(\cos x)(\sin x)}=$$
$$=\frac{2\cos^2x+2\sin x\cos x}{(\cos x-\sin x)(\cos x+\sin x)}=$$
$$=\frac{2\cos x(\cos x+\sin x)}{(\cos x-\sin x)(\cos x+\sin x)}=$$
$$=\frac{2\cos x}{\cos x-\sin x}=$$
$$=\frac{2\cos x}{\cos x\left(1-\dfrac{\sin x}{\cos x}\right)}=\text{, pois }\forall x\in\left]0,\frac{\pi}{2}\right[,\ \cos\ x\neq 0.$$
$$=\frac{2}{1-\tan x}$$

Logo, $f(2x)=\dfrac{2}{1-\tan x}$.

16. $\sin\left(\dfrac{\pi}{2}-\theta\right)\sin(4\theta)+\sin\left(\dfrac{3\pi}{2}+4\theta\right)\sin(\pi-\theta)=\dfrac{2}{3}\Leftrightarrow$

$$\Leftrightarrow\cos(\theta)\sin(4\theta)+(-\cos(4\theta))(\sin\theta)=\frac{2}{32}\Leftrightarrow$$
$$\Leftrightarrow\cos\theta\sin(4\theta)-\cos(4\theta)\sin\theta=\frac{2}{3}\Leftrightarrow$$
$$\Leftrightarrow\sin(4\theta)\cos\theta-\sin\theta\cos(4\theta)=\frac{2}{3}\Leftrightarrow$$
$$\Leftrightarrow\sin(4\theta-\theta)=\frac{2}{3}\Leftrightarrow\sin(3\theta)=\frac{2}{3}$$

Por outro lado:

$$\cos(6\theta)=\cos^2(3\theta)-\sin^2(3\theta)=$$
$$=1-\sin^2(3\theta)-\sin^2(3\theta)=$$
$$=1-2\sin^2(3\theta)=1-2\times\left(\frac{2}{3}\right)^2=1-\frac{8}{9}=\frac{1}{9}$$

17. $\tan\beta+\dfrac{1}{\tan\beta}=3\Leftrightarrow\dfrac{\sin\beta}{\cos\beta}+\dfrac{\cos\beta}{\sin\beta}=3\Leftrightarrow$

$$\Leftrightarrow\frac{\sin^2\beta+\cos^2\beta}{\cos\beta\sin\beta}=3\Leftrightarrow$$
$$\Leftrightarrow\frac{1}{\cos\beta\sin\beta}=3\Leftrightarrow$$
$$\Leftrightarrow 3\cos\beta\sin\beta=1\wedge\cos\beta\sin\beta\neq 0\Leftrightarrow$$
$$\Leftrightarrow\cos\beta\sin\beta=\frac{1}{3}\Leftrightarrow 2\cos\beta\sin\beta=\frac{2}{3}\Leftrightarrow$$
$$\Leftrightarrow\sin(2\beta)=\frac{2}{3}$$

18.1. $\tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=$

$$=\frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

- O triângulo $[ADC]$ é isósceles e retângulo em A

  Logo, $\alpha=\dfrac{\pi}{4}$ pelo que $\cos\alpha=\sin\alpha=\dfrac{\sqrt{2}}{2}$.

- $\overline{AB}=1+1+\sqrt{3}=2+\sqrt{3}$

  $\overline{BC}^2=1^2+\left(2+\sqrt{3}\right)^2=1+4+4\sqrt{3}+3=8+4\sqrt{3}$

  Como $\overline{BC}>0$, vem $\overline{BC}=\sqrt{8+4\sqrt{3}}$.

$$\cos\beta=\frac{\overline{AB}}{\overline{BC}}=\frac{2\sqrt{3}}{\sqrt{8+4\sqrt{3}}}\ ;\ \sin\beta=\frac{\overline{AC}}{\overline{BC}}=\frac{1}{\sqrt{8+4\sqrt{3}}}$$

$$\tan(\alpha+\beta)=\frac{\dfrac{\sqrt{2}}{2}\times\dfrac{2+\sqrt{3}}{\sqrt{8+4\sqrt{3}}}+\dfrac{\sqrt{2}}{2}\times\dfrac{1}{\sqrt{8+4\sqrt{3}}}}{\dfrac{\sqrt{2}}{2}\times\dfrac{2+\sqrt{3}}{\sqrt{8+4\sqrt{3}}}-\dfrac{\sqrt{2}}{2}\times\dfrac{1}{\sqrt{8+4\sqrt{3}}}}=$$
$$=\frac{\dfrac{\sqrt{2}}{2}\times\dfrac{2+\sqrt{3}+1}{\sqrt{8+4\sqrt{3}}}}{\dfrac{\sqrt{2}}{2}\times\dfrac{2+\sqrt{3}-1}{\sqrt{8+4\sqrt{3}}}}=\frac{3+\sqrt{3}}{1+\sqrt{3}}=$$
$$=\frac{\left(3+\sqrt{3}\right)\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}=\frac{3-3\sqrt{3}+\sqrt{3}-3}{1-3}=\frac{-2\sqrt{3}}{-2}=\sqrt{3}$$

18.2. $\cos(2\beta)=\cos^2\beta-\sin^2\beta=$

$$=\left(\frac{2+\sqrt{3}}{\sqrt{8+4\sqrt{3}}}\right)^2-\left(\frac{1}{\sqrt{8+4\sqrt{3}}}\right)^2=$$
$$=\frac{\left(2+\sqrt{3}\right)^2}{8+4\sqrt{3}}-\frac{1}{8+4\sqrt{3}}=$$

$$= \frac{4+4\sqrt{3}+3-1}{4\left(2+\sqrt{3}\right)} = \frac{\left(6+4\sqrt{3}\right)\left(2-\sqrt{3}\right)}{4\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)} =$$

$$= \frac{12-6\sqrt{3}+8\sqrt{3}-12}{4(4-3)} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

Como $\cos(2\beta) = \frac{\sqrt{3}}{2}$ e 2β é um ângulo agudo, então $2\beta = 30^\circ$, ou seja, $\beta = 15^\circ$.

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19.1. $\lim\limits_{x\to0}\frac{\sin(4x)}{x} = \lim\limits_{x\to0}\left[\frac{\sin(4x)}{4x}\times 4\right] = 4\times\lim\limits_{x\to0}\frac{\sin(4x)}{4x} =$

$= 4\times\lim\limits_{x\to0}\frac{\sin(4x)}{4x} = 4\times\lim\limits_{y\to0}\frac{\sin(y)}{y} =$

$= 4\times1 = 4$

| $y = 4x$; Se $x\to0$, $y\to0$

19.2. $\lim\limits_{x\to0}\frac{2x}{\sin(3x)} = 2\lim\limits_{x\to0}\frac{x}{\sin(3x)} = 2\lim\limits_{x\to0}\left[\frac{3x}{\sin(3x)}\times\frac{1}{3}\right] =$

$= \frac{2}{3}\lim\limits_{x\to0}\frac{3x}{\sin(3x)} = \frac{2}{3}\times\frac{1}{\lim\limits_{x\to0}\frac{\sin(3x)}{3x}} =$

$= \frac{2}{3}\times\frac{1}{\lim\limits_{y\to0}\frac{\sin(y)}{y}} = \frac{2}{3}\times\frac{1}{1} = \frac{2}{3}$

| $y = 3x$; Se $x\to0$, $y\to0$

19.3. $\lim\limits_{x\to0}\frac{\sin(x)+\sin(5x)}{3x} = \frac{1}{3}\lim\limits_{x\to0}\frac{\sin(x)+\sin(5x)}{x} =$

$= \frac{1}{3}\left[\lim\limits_{x\to0}\frac{\sin(x)}{x} + \lim\limits_{x\to0}\frac{\sin(5x)}{x}\right] =$

$= \frac{1}{3}\left[1 + 5\lim\limits_{x\to0}\frac{\sin(5x)}{5x}\right] =$

$= \frac{1}{5} + \frac{5}{3}\lim\limits_{x\to0}\frac{\sin(5x)}{5x} =$

$= \frac{1}{3} + \frac{5}{3}\lim\limits_{y\to0}\frac{\sin(y)}{y} =$

$= \frac{1}{3} + \frac{5}{3}\times1 = \frac{1}{3} + \frac{5}{3} = 2$

| $y = 5x$; Se $x\to0$, $y\to0$

19.4. $\lim\limits_{x\to0}\frac{\sin(6x)}{\sin(5x)} = \lim\limits_{x\to0}\left[\frac{\sin(6x)}{6x}\times\frac{5x}{\sin(5x)}\times\frac{6}{5}\right] =$

$= \frac{6}{5}\lim\limits_{x\to0}\frac{\sin(6x)}{6x}\times\lim\limits_{x\to0}\frac{5x}{\sin(5x)} =$

$= \frac{6}{5}\lim\limits_{x\to0}\frac{\sin(6x)}{6x}\times\frac{1}{\lim\limits_{5x\to0}\frac{\sin(5x)}{5x}} =$

$= \frac{6}{5}\lim\limits_{u\to0}\frac{\sin u}{u}\times\frac{1}{\lim\limits_{v\to0}\frac{\sin v}{v}} =$

$= \frac{6}{5}\times1\times\frac{1}{1} = \frac{6}{5}$

| $u = 6x$; Se $x\to0$, $u\to0$; $v = 5x$; Se $x\to0$, $v\to0$

19.5. $\lim\limits_{x\to0}\frac{3x+x^2}{\sin x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to0}\frac{x(3+x)}{\sin x} = \lim\limits_{x\to0}\frac{x}{\sin x}\times\lim\limits_{x\to0}(3+x) =$

$= \frac{1}{\lim\limits_{x\to0}\frac{\sin x}{x}}\times3 = \frac{1}{1}\times3 = 3$

19.6. $\lim\limits_{x\to0}\frac{\tan x}{x})\lim\limits_{x\to0}\frac{\frac{\sin x}{\cos x}}{x} = \lim\limits_{x\to0}\frac{\sin x}{x\cos x} =$

$= \lim\limits_{x\to0}\frac{\sin x}{x}\times\lim\limits_{x\to0}\frac{1}{\cos x} = 1\times\frac{1}{1} = 1$

19.7. $\lim\limits_{x\to0}\frac{\tan(3x)}{\sin(4x)} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to0}\frac{\frac{\sin(3x)}{\cos(3x)}}{\sin(4x)} = \lim\limits_{x\to0}\frac{\sin(3x)}{\sin(4x)\cos(3x)} =$

$= \lim\limits_{x\to0}\frac{\sin(3x)}{3x}\times\lim\limits_{x\to0}\frac{4x}{\sin(4x)}\times\lim\limits_{x\to0}\frac{1}{\cos(3x)}\times\frac{3}{4} =$

$= \lim\limits_{\frac{3}{4}x\to0}\frac{\sin(3x)}{3x}\times\frac{1}{\lim\limits_{x\to0}\frac{\sin(4x)}{4x}}\times\frac{1}{1} =$

$= \frac{3}{4}\times1\times\frac{1}{1} = \frac{3}{4}$

| $u = 3x$; Se $x\to0$, $u\to0$
| $v = 4x$; Se $x\to0$, $v\to0$

19.8. $\lim\limits_{x\to0}\frac{\sin x}{3x-\tan(2x)} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to0}\frac{\sin x}{3x-\frac{\sin(2x)}{\cos(2x)}} = \lim\limits_{x\to0}\frac{\frac{\sin x}{x}}{\frac{3x}{x}-\frac{\sin(2x)}{x\cos(2x)}} =$

$= \frac{\lim\limits_{x\to0}\frac{\sin x}{x}}{3-\lim\limits_{x\to0}\frac{\sin(2x)}{x\cos(2x)}} = \frac{1}{3-2\lim\limits_{x\to0}\frac{\sin(2x)}{2x}\times\lim\limits_{x\to0}\frac{1}{\cos(2x)}} =$

$= \frac{1}{3-2\lim\limits_{y\to0}\frac{\sin y}{y}\times\frac{1}{1}} =$

| $y = 2x$; Se $x\to0$, $y\to0$

$= \frac{1}{3-2\times1\times1} = 1$

19.9. $\lim\limits_{x\to0}\frac{\sin^2 x}{x} = \lim\limits_{x\to0}\frac{\sin x}{x}\times\lim\limits_{x\to0}\sin x = 1\times0 = 0$

19.10. $\lim\limits_{x\to0}\frac{\cos x-1}{x} \overset{\left(\frac{0}{0}\right)}{=} -\lim\limits_{x\to0}\frac{1-\cos x}{x} =$

$= -\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = -\lim\limits_{x\to0}\frac{1-\cos^2 x}{x(1+\cos x)} =$

$= -\lim\limits_{x\to0}\frac{\sin^2 x}{x(1+\cos x)} = -\lim\limits_{x\to0}\frac{\sin x}{x}\times\lim\limits_{x\to0}\frac{\sin x}{1+\cos x} = 0$

20.11. $\lim\limits_{x\to0}\frac{1-\cos x}{x^2} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} =$

$= \lim\limits_{x\to0}\frac{1-\cos^2 x}{x^2(1+\cos x)} =$

$= \lim\limits_{x\to0}\frac{\sin^2 x}{x^2(1+\cos(x))} = \lim\limits_{x\to0}\frac{\sin^2 x}{x^2}\times\lim\limits_{x\to0}\frac{1}{1+\cos x} =$

$= \left(\lim\limits_{x\to0}\frac{\sin x}{x}\right)^2\times\frac{1}{1+1} = 1^2\times\frac{1}{2} = \frac{1}{2}$

19.12. $\lim\limits_{x\to0}\frac{\sin^2 x\cos^2 x}{1-\cos x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to0}\frac{\sin^2 x\cos^2 x(1+\cos x)}{(1-\cos x)(1+\cos x)} =$

$= \lim\limits_{x\to0}\frac{\sin^2 x\cos^2 x(1+\cos x)}{1-\cos^2 x} =$

$= \lim\limits_{x\to0}\frac{\sin^2 x\cos^2 x(1+\cos x)}{\sin^2 x} =$

$= \lim\limits_{x\to0}\left[\cos^2 x(1+\cos x)\right] = 1\times(1+1) = 2$

19.13. $\lim\limits_{x\to-\infty}\frac{\sin x}{x}=\lim\limits_{x\to-\infty}\left(\sin x\times\frac{1}{x}\right)=0$

dado que $\forall x\in\mathbb{R}, -1\le\sin x\le 1$, $\lim\limits_{x\to-\infty}\frac{1}{x}=0$ e o limite de uma função limitada por uma função de limite nulo é igual a 0.

19.14. $\lim\limits_{x\to\frac{\pi}{2}}\frac{\tan(x)}{x-\frac{\pi}{2}}$, temos de calcular os limites laterias:

$\lim\limits_{x\to\frac{\pi}{2}^-}\frac{\tan(x)}{x-\frac{\pi}{2}}=\frac{+\infty}{0^-}=-\infty$ e $\lim\limits_{x\to\frac{\pi}{2}^+}\frac{\tan(x)}{x-\frac{\pi}{2}}=\frac{-\infty}{0^+}=-\infty$

Portanto, $\lim\limits_{x\to\frac{\pi}{2}}\frac{\tan(x)}{x-\frac{\pi}{2}}=-\infty$

19.15. $\lim\limits_{x\to0}\frac{x^2-x}{\sin x}=\lim\limits_{x\to0}\frac{x(x-1)}{\sin x}=$

$=\lim\limits_{x\to0}\frac{x}{\sin x}\times\lim\limits_{x\to0}(x-1)=\frac{1}{\lim\limits_{x\to0}\frac{\sin x}{x}}\times(-1)=\frac{1}{1}\times(-1)=-1$

19.16. $\lim\limits_{x\to0}\frac{\sin(-x)}{\sin(4x)}=-\lim\limits_{x\to0}\frac{\sin x}{x}\times\lim\limits_{x\to0}\frac{x}{\sin(4x)}=$

$=-1\times\frac{1}{4}\times\lim\limits_{x\to0}\frac{4x}{\sin(4x)}=$

$=-\frac{1}{4}\times\frac{1}{\lim\limits_{y\to0}\frac{\sin y}{y}}=-\frac{1}{4}$ | $y=4x$; Se $x\to0$, $y\to0$

19.17. $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos(x)}{\frac{\pi}{2}-x}\overset{\left(\frac{0}{0}\right)}{=}$ | $y=x-\frac{\pi}{2}\Leftrightarrow x=\frac{\pi}{2}+y$; Se $x\to\frac{\pi}{2}$, $y\to0$

$=\lim\limits_{y\to0}\frac{\cos\left(\frac{\pi}{2}+y\right)}{-y}=$

$=\lim\limits_{y\to0}\frac{-\sin}{-y}=\lim\limits_{y\to0}\frac{\sin y}{y}=1$

19.18. $\lim\limits_{x\to3}\frac{\sin(x-3)}{x^2-9}\overset{\left(\frac{0}{0}\right)}{=}\lim\limits_{x\to3}\frac{\sin(x-3)}{(x-3)(x+3)}=$

$=\lim\limits_{x\to3}\frac{1}{x+3}\times\lim\limits_{x\to3}\frac{\sin(x-3)}{x-3}$

$=\frac{1}{3+3}\times\lim\limits_{y\to0}\frac{\sin y}{y}=\frac{1}{6}$ | $y=x-3$; Se $x\to3$, $y\to0$

19.19. $\lim\limits_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\cos(2x)}\overset{\left(\frac{0}{0}\right)}{=}\lim\limits_{x\to\frac{\pi}{4}}\frac{1-\frac{\sin x}{\cos(x)}}{\cos^2x-\sin^2x}=$

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{\frac{\cos x-\sin x}{\cos x}}{\cos^2x-\sin^2x}=\lim\limits_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos x\left(\cos^2x-\sin^2x\right)}=$

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{1}{\cos(x)(\cos x+\sin x)}=$

$=\frac{1}{\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)}=\frac{1}{\frac{1}{2}+\frac{1}{2}}=\frac{1}{1}=1$

19.20. $\lim\limits_{x\to0}\frac{\cos\left(x-\frac{\pi}{2}\right)}{\sin(2x)}\overset{\left(\frac{0}{0}\right)}{=}$

$=\lim\limits_{x\to0}\frac{\sin x}{\sin(2x)}=$

$=\lim\limits_{x\to0}\frac{\sin x}{2\sin x\cos x}=\lim\limits_{x\to0}\frac{1}{2\cos x}=\frac{1}{2}$

19.21. $\lim\limits_{x\to+\infty}\left[x\sin\left(\frac{1}{x}\right)\right]=$ | $y=\frac{1}{x}\Leftrightarrow x=\frac{1}{y}$; Se $x\to+\infty$, $y\to0$

$=\lim\limits_{y\to0}\left(\frac{1}{y}\times\sin y\right)=\lim\limits_{y\to0}\frac{\sin(y)}{y}=1$

19.22. $\lim\limits_{x\to0}\frac{|x|}{\sin x}\overset{\left(\frac{0}{0}\right)}{=}$, calculemos os limites laterais:

$\lim\limits_{x\to0^+}\frac{|x|}{\sin(x)}=\lim\limits_{x\to0^+}\frac{x}{\sin x}=\frac{1}{\lim\limits_{x\to0^+}\frac{\sin x}{x}}=\frac{1}{1}=1$

$\lim\limits_{x\to0^-}\frac{|x|}{\sin x}=\lim\limits_{x\to0^-}\frac{-x}{\sin x}=-\frac{1}{\lim\limits_{x\to0^-}\frac{\sin x}{x}}=-\frac{1}{1}=-1$

Como $\lim\limits_{x\to0^+}\frac{|x|}{\sin x}\ne\lim\limits_{x\to0^-}\frac{|x|}{\sin x}$, não existe $\lim\limits_{x\to0}\frac{|x|}{\sin x}$.

19.23. $\lim\limits_{x\to0}\frac{\tan(x)-\sin(x)}{x^3}=\lim\limits_{x\to0}\frac{\frac{\sin(x)}{\cos(x)}-\sin(x)}{x^3}=$

$=\lim\limits_{x\to0}\frac{\frac{\sin(x)-\sin(x)\cos x}{\cos(x)}}{x^3}=\lim\limits_{x\to0}\frac{\sin x-\sin x\cos x}{x^3\cos x}=$

$=\lim\limits_{x\to0}\frac{\sin x(1-\cos x)}{x^3\cos x}=$

$=\lim\limits_{x\to0}\frac{\sin x}{x}\times\lim\limits_{x\to0}\frac{1-\cos x}{x^2}\times\lim\limits_{x\to0}\frac{1}{\cos x}=$

$=1\times\lim\limits_{x\to0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}\times1=$

$=\lim\limits_{x\to0}\frac{1-\cos^2x}{x^2}\times\lim\limits_{x\to0}\frac{1}{1+\cos x}=$

$=\lim\limits_{x\to0}\frac{\sin x}{x^2}\times\frac{1}{1+1}=\left(\lim\frac{\sin x}{x}\right)^2\times\frac{1}{2}=\frac{1}{2}$

19.24. $\lim\limits_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos x}=\lim\limits_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos^2x-\sin^2x}=$

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{(\cos x-\sin x)(\cos x+\sin x)}=$

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{1}{\cos x+\sin x}=\frac{1}{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$

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20. Assíntotas verticais

A função é contínua em $\left]-\frac{\pi}{2},\frac{3\pi}{2}\right[$ pois é definida pela soma e quociente de duas funções contínuas (função polinomial).

Assim, as únicas possíveis assíntotas verticais são as de equações $x=-\frac{\pi}{2}$ e $x=\frac{3\pi}{2}$.

$$\lim_{x\to-\frac{\pi}{2}^+} f(x)=\lim_{x\to-\frac{\pi}{2}^+}\frac{\sin x}{1+\sin x}=\frac{-1}{0^+}=-\infty$$

A reta de equação $x=-\frac{\pi}{2}$ é uma assíntota ao gráfico de f.

$$\lim_{x\to\frac{3\pi}{2}^-} f(x)=\lim_{x\to\frac{3\pi}{2}^-}\frac{\sin x}{1+\sin x}=\frac{-1}{0^+}=-\infty$$

A reta de equação $x=\frac{3\pi}{2}$ é uma assíntota ao gráfico de f

Assíntotas não verticais

O gráfico de f não tem assíntoas não verticais, uma vez que o domínio é um conjunto limitado.

21.1. $D_g=\{x\in\mathbb{R}:\sin(3x)\neq 0\}=\{x\in\mathbb{R}:3x\neq k\pi,\ k\in\mathbb{Z}\}=$

$=\left\{x\in\mathbb{R}:x\neq\frac{k\pi}{3},\ k\in\mathbb{Z}\right\}\Leftrightarrow D_g=\mathbb{R}\setminus\left\{\frac{k\pi}{3},\ k\in\mathbb{Z}\right\}$

21.2. $g(x)=0\Leftrightarrow\frac{x}{\sin(3x)}=0\Leftrightarrow x=0\wedge x\in D_g\Leftrightarrow x\in\varnothing$

A função g não tem zeros.

22. A função h é contínua no intervalo $]0,+\infty[$ pois é aí definida pela soma e quociente de funções contínuas (função polinomial e função seno).

A função h é contínua no intervalo $]-\infty,0[$ pois é aí definida pela diferença e quociente de funções contínuas (função seno e funções polinomiais).

Investiguemos se a função h é contínua em $x=0$.

$$\lim_{x\to0^+}h(x)=\lim_{x\to0^+}\frac{x+\sin x}{x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0^+}\frac{x}{x}+\lim_{x\to0^+}\frac{\sin x}{x}=1+1=2$$

$$\lim_{x\to0^-}h(x)=\lim_{x\to0^-}\frac{1-\cos x}{x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0^-}\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}=$$

$$=\lim_{x\to0^-}\frac{1-\cos^2 x}{x(1+\cos x)}=$$

$$=\lim_{x\to0^-}\frac{\sin^2 x}{x(1+\cos x)}=\lim_{x\to0^-}\frac{\sin x}{x}\times\lim_{x\to0^-}\frac{\sin x}{1+\cos x}=0$$

Como $\lim_{x\to0^+}h(x)\neq\lim_{x\to0^-}h(x)$, não existe $\lim_{x\to0}$ pelo que a função h não é contínua em $x=0$.

Conclusão: A função h é contínua em $\mathbb{R}\setminus\{0\}$.

23.1. A função f é contínua no intervalo $]-\infty,0[$ pois é aí definida pela composta e quociente de duas funções contínuas (função seno e funções polinomiais).

A função f é contínua no intervalo $]0,+\infty[$ pois é constante nesse intervalo. No ponto $x=0$:

$f(0)=\lim_{x\to0^+}f(x)=a$

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\frac{\sin(2x)}{4x}=$$

$$=\lim_{x\to0^-}\left[\frac{\sin(2x)}{2x}\times\frac{1}{2}\right]=\frac{1}{2}\lim_{x\to0^-}\frac{\sin(2x)}{2x}$$

$$=\frac{1}{2}\lim_{x\to0^-}\frac{\sin y}{y}=\frac{1}{2}\times1=\frac{1}{2}$$

$y=2x$; Se $x\to0^-$, $y\to0^-$

Assim, f é com tínua em $\mathbb{R}$ se e só se $a=\frac{1}{2}$.

23.2. Assíntotas verticais

A função f é contínua em $\mathbb{R}\setminus\{0\}$.

Em $x=0$:

$\lim_{x\to0^+}f(x)=a$; $\lim_{x\to0^-}f(x)=\frac{1}{2}$

Logo, o gráfico de f não tem assíntotas verticais.

Assíntotas não verticais

Em $-\infty$:

$\forall x\in\mathbb{R}$, $-1\leq\sin(2x)\leq1$ e $\lim_{x\to-\infty}\frac{1}{4x}=0$

Logo, $\lim_{x\to-\infty}f(x)=0$ por ser o produto de uma função limitada por uma função de limite nulo.

A a reta de equação $y=0$ é assíntota ao gráfico de f em $-\infty$.

$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}a=a$

Logo, a reta de equação $y=a$ é uma assíntota ao gráfico de f em $+\infty$.

24. A função h é contínua em $x=3$ quando e apenas quando $\lim_{x\to3^-}h(x)=\lim_{x\to3^+}h(x)=h(3)$.

$$h(3)=\lim_{x\to3^+}h(x)=k+\cos\frac{\pi}{3}=k+\frac{1}{2}$$

$$\lim_{x\to3^-}h(x)=\lim_{x\to3^-}\frac{3x^2-8x-3}{\sin(3-x)}\overset{\left(\frac{0}{0}\right)}{=}$$

Portanto, $3x^2-8x-3=(x-3)(3x+1)$.

Recorrendo a uma regra de Ruffini, tem-se:

	3	–8	–3
3		9	3
	3	1	0

$$\lim_{x\to3^-}\frac{3x^2-8x-3}{\sin(3-x)}=\lim_{x\to3^-}\frac{(x-3)(3x+1)}{\sin(3-x)}=$$

$$=-\lim_{x\to3^-}\frac{3-x}{\sin(3-x)}\times\lim_{x\to3^-}(3x+1)=$$

$y=3-x$; Se $x\to3^-$, $y\to0^+$

$$=-\lim_{y\to0^+}\frac{y}{\sin y}\times(9+1)=-\frac{1}{\lim_{y\to0}\frac{\sin y}{y}}\times10=-10$$

Logo, h é contínua em $x=3$ se

$k+\frac{1}{2}=-10\Leftrightarrow k=-10-\frac{1}{2}\Leftrightarrow k=-\frac{21}{2}$.

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25.1. $\lim_{x\to0^-}g(x)=\lim_{x\to0^-}\frac{3x^2-7x+2}{x-6}=\frac{2}{-6}=-\frac{1}{3}$

$g(0)=\lim_{x\to0^-}g(x)=-\frac{1}{3}$

$$\lim_{x\to0^+}g(x)=\lim_{x\to0^+}\frac{x-\sin(3x)}{x+\sin(5x)}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0^+}\frac{x\left(1-\frac{\sin(3x)}{x}\right)}{x\left(1+\frac{\sin(5x)}{x}\right)}=$$

$$=\lim_{x\to0^+}\frac{1-\frac{\sin(3x)}{3x}\times3}{1+\frac{\sin(5x)}{5x}\times5}=\frac{1-3\lim_{u\to0}\frac{\sin u}{u}}{1+\lim_{v\to0}\frac{\sin v}{v}}=$$

$u=3x$; Se $x\to0$, $u\to0$

$v=5x$; Se $x\to0$, $v\to0$

$$=\frac{1-3\times1}{1+5\times1}=\frac{-2}{6}=-\frac{1}{3}$$

Como $\lim_{x\to 0^-} g(x) = g(0) = \lim_{x\to 0^+} g(x) = -\frac{1}{3}$, então existe $\lim_{x\to 0} g(x)$ e a função g é contínua no ponto $x = 0$.

25.2. $g(x) = 1 \Leftrightarrow$

$$\Leftrightarrow \left(\frac{3x^2-7x+2}{x-6}=1 \wedge x\le 0\right) \vee \left(\frac{x-\sin(3x)}{x+\sin(5x)}=1 \wedge x>0\right) \Leftrightarrow$$

$$\Leftrightarrow \left(3x^2-7x+2 = x-6 \wedge x\le 0\right)\vee$$

$$\vee\left(x-\sin(3x) = x+\sin(5x)\wedge x>0\right)\Leftrightarrow$$

$$\Leftrightarrow\left(3x^2-8x+8=0\wedge x\le 0\right)\vee\left(-\sin(3x)=\sin(5x)\wedge x>0\right)\Leftrightarrow$$

$$\Leftrightarrow\left(x=\frac{8\pm\sqrt{-32}}{2\times 3}\wedge x\le 0\right)\vee\left(\sin(-3x)=\sin(5x)\wedge x>0\right)\Leftrightarrow$$

$$\Leftrightarrow x\in\varnothing \vee -3x=5x+2k\pi \vee -3x=\pi-5x+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$$

$$\Leftrightarrow x\in\varnothing\vee\left(-8x=2k\pi\vee 2x=\pi+2k\pi,\ k\in\mathbb{Z}\right)\wedge x>0\Leftrightarrow$$

$$\Leftrightarrow\left(x=-\frac{k\pi}{4}\vee x=\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}\right)\wedge x>0\Leftrightarrow$$

$$\Leftrightarrow x=-\frac{k\pi}{4},\ k\in\mathbb{Z}^-\vee x=\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}_0^+$$

Existem, portanto, infinitos pontos de interseção da reta r com o gráfico de g.

26.1. $D_h = \{x\in\mathbb{R}: 1-2\cos x\neq 0\}$

$$1-2\cos x = 0\Leftrightarrow 2\cos x = 1\Leftrightarrow \cos x=\frac{1}{2}\Leftrightarrow\cos x=\cos\frac{\pi}{3}\Leftrightarrow$$

$$\Leftrightarrow x=\frac{\pi}{3}+2k\pi\vee x=-\frac{\pi}{3}+2k\pi, k\in\mathbb{Z}$$

Portanto, $D_h = \mathbb{R}\setminus\left\{x: x=\frac{\pi}{3}+2k\pi\vee x=-\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}\right\}$

26.2. $\forall x\in D_h, h(x)=\frac{\sin(3x)}{1-2\cos x}=\frac{\sin(2x+x)}{1-2\cos x}=$

$$=\frac{\sin(2x)\cos x+\sin x\cos(2x)}{1-2\cos x}=$$

$$=\frac{2\sin x\cos x\cos x+\sin x\left(\cos^2 x-\sin^2 x\right)}{1-2\cos x}=$$

$$=\frac{2\sin x\cos^2 x+\sin x\left[\cos^2 x-\left(1-\cos^2 x\right)\right]}{1-2\cos x}=$$

$$=\frac{2\sin x\cos^2 x+\sin x\left(2\cos^2 x-1\right)}{1-2\cos x}=$$

$$=\frac{2\sin x\cos^2 x+2\sin x\cos^2 x-\sin x}{1-2\cos x}=$$

$$=\frac{4\sin x\cos^2 x-\sin x}{1-2\cos x}=\frac{\sin x\left(4\cos^2 x-1\right)}{1-2\cos x}=$$

$$=\frac{\sin x(2\cos x-1)(2\cos x+1)}{1-2\cos x}=$$

$$=\frac{\sin x(2\cos x-1)(2\cos x+1)}{-(2\cos x-1)}=\frac{\sin x(2\cos x+1)}{-1}=$$

$$=-\sin x(1+2\cos x)$$

26.3. $\lim_{x\to\frac{\pi}{3}} h(x)=\lim_{x\to\frac{\pi}{3}}\left[-\sin x(1+2\cos x)\right]=$

$$=-\sin\left(\frac{\pi}{3}\right)\left[1+2\cos\left(\frac{\pi}{3}\right)\right]=$$

$$=-\frac{\sqrt{3}}{2}\left(1+2\times\frac{1}{2}\right)=-\sqrt{3}$$

27.1. Cálculo auxiliar

$$2\sin^2 x-3\sin x+1=0\Leftrightarrow\sin x=\frac{3\pm\sqrt{(-3)^2-4\times 2\times 1}}{2\times 2}$$

$$\Leftrightarrow\sin x=\frac{3\pm 1}{4}\Leftrightarrow\sin x=1\vee\sin=\frac{1}{2}$$

$$2\sin^2 x+\sin x-1=0\Leftrightarrow\sin x=\frac{-1\pm\sqrt{1^2-4\times 2\times(-1)}}{2\times 2}\Leftrightarrow$$

$$\Leftrightarrow\sin x=\frac{-1\pm 3}{4}\Leftrightarrow\sin x=-1\vee\sin x=\frac{1}{2}$$

$$\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x-3\sin x+1}{2\sin^2 x+\sin x-1}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to\frac{\pi}{6}}\frac{2(\sin x-1)\left(\sin x-\frac{1}{2}\right)}{2(\sin x+1)\left(\sin x-\frac{1}{2}\right)}=$$

$$=\lim_{x\to\frac{\pi}{6}}\frac{\sin x-1}{\sin x+1}=\frac{\sin\frac{\pi}{6}-1}{\sin\frac{\pi}{6}+1}=\frac{\frac{1}{2}-1}{\frac{1}{2}+1}=\frac{-\frac{1}{2}}{\frac{3}{2}}=-\frac{1}{3}$$

27.2. $\lim_{x\to 1}\frac{1-x^2}{\sin(\pi x)}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 1}\frac{(1-x)(1+x)}{\sin(\pi x)}=\lim_{x\to 1}\frac{(1-x)(1+x)}{\sin(\pi-\pi x)}=$

$$=-\lim_{x\to 1}(1+x)\times\lim_{x\to 1}\frac{x-1}{\sin(\pi x)}=$$

$$=-(1+1)\times\lim_{y\to 0}\frac{y}{\sin\left[\pi(y+1)\right]}=\qquad \left|\begin{array}{l}y=x-1\Leftrightarrow x=y+1\\ \text{Se } x\to 1, y\to 0\end{array}\right.$$

$$=-2\times\lim_{y\to 0}\frac{y}{\sin(\pi y+\pi)}=-2\times\lim_{y\to 0}\frac{y}{-\sin(\pi y)}=$$

$$=2\times\frac{1}{\pi}\times\lim_{y\to 0}\frac{\pi y}{\sin(\pi y)}=\frac{2}{\pi}\times\frac{1}{\lim_{u\to 0}\frac{\sin u}{u}}=\qquad\left|\begin{array}{l}u=\pi y\\ \text{Se } y\to 0, u\to 0\end{array}\right.$$

$$=\frac{2}{\pi}\times\frac{1}{1}=\frac{2}{\pi}$$

27.3. $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{1-\tan x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{1-\frac{\sin x}{\cos x}}=$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\frac{\cos x-\sin x}{\cos x}}=$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)\cos x}{\cos x-\sin x}=\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)\cos x}{-(\sin x-\cos x)}=$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\cos x}{-1}=\frac{\cos\frac{\pi}{4}}{-1}=\frac{\frac{\sqrt{2}}{2}}{-1}=-\frac{\sqrt{2}}{2}$$

27.4. $\lim_{x\to 1}\frac{\sin(x-1)}{\sqrt{2x-1}-1}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 1}\frac{\sin(x-1)\left(\sqrt{2x-1}+1\right)}{\left(\sqrt{2x-1}-1\right)\left(\sqrt{2x-1}+1\right)}=$

$$=\lim_{x\to 1}\frac{\sin(x-1)\left(\sqrt{2x-1}+1\right)}{\left(\sqrt{2x-1}\right)^2-1}=$$

$$=\lim_{x\to 1}\frac{\sin(x-1)\left(\sqrt{2x-1}+1\right)}{2x-1-1}=$$

$$=\lim_{x\to 1}\frac{\sin(x-1)\left(\sqrt{2x-1}+1\right)}{2(x-1)}=$$

$$= \frac{1}{2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1}\times\lim_{x\to 1}\left(\sqrt{2x-1}+1\right) =$$

$$= \frac{1}{2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1}\times 2 =$$

$$= \frac{1}{2}\times\lim_{y\to 0}\frac{\sin y}{y} = 1 \qquad \left| \begin{array}{l} y = \pi - 1 \\ \text{Se } x \to 1, y \to 0 \end{array}\right.$$

27.5. $\lim\limits_{x\to\frac{\pi}{4}}\frac{\sin(2x)-\cos(2x)-1}{\cos x-\sin x} \overset{\left(\frac{0}{0}\right)}{=}$

$$= \lim_{x\to\frac{\pi}{4}}\frac{2\sin x\cos x-\left(\cos^2 x-\sin^2 x\right)-1}{\cos x-\sin x} =$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{2\sin x\cos x-\left(\cos^2 x-\left(1-\cos^2 x\right)\right)-1}{\cos x-\sin x} =$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{2\sin x\cos x-2\cos^2 x}{\cos x-\sin x} =$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{-2\cos x(\cos x-\sin x)}{\cos x-\sin x} = \lim_{x\to\frac{\pi}{4}}(-2\cos x) =$$

$$= -2\cos\frac{\pi}{4} = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$$

27.6. $\lim\limits_{x\to 0}\frac{x}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \overset{\left(\frac{0}{0}\right)}{=}$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)} =$$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2} =$$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}{1+\sin x-(1-\sin x)} =$$

$$= \lim_{x\to 0}\frac{x\left(\sqrt{1+\sin x}\sqrt{1-\sin x}\right)}{2\sin x} =$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{x}{\sin x}\times\lim_{x\to 0}\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right) =$$

$$= \frac{1}{2}\times\frac{1}{\lim\limits_{x\to 0}\frac{\sin x}{x}}\times 2 = \frac{1}{2}\times\frac{1}{1}\times 2 = 1$$

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28. $D_f = D_g = \mathbb{R}$

$$f(x) = \sin(3x) = \sin(x+2x) =$$

$$= \sin x\cos(2x)+\sin(2x)\cos x =$$

$$= \sin x\left(\cos^2 x-\sin^2 x\right)+2\sin x\cos x\cos x =$$

$$= \sin x\left(1-2\sin^2 x\right)+2\sin x\cos^2 x =$$

$$= \sin x-2\sin^3 x+2\sin x\cos^2 x =$$

$$= \sin x-2\sin^3 x+2\sin x\left(1-\sin^2 x\right) =$$

$$= \sin x-2\sin^3 x+2\sin x-2\sin^3 x =$$

$$= 3\sin x-4\sin^3 x$$

Como $D_f = D_g \wedge f(x) = g(x), \forall x \in D_f$ temos que $f = g$.

29.1.
- $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\frac{x^2+2x}{x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0^+}\frac{x(x+2)}{x} = 2$
- $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\frac{\sin x^2}{1-\cos x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0^-}\frac{\sin x^2(1+\cos x)}{(1-\cos x)(1+\cos x)} =$

$$= \lim_{x\to 0^-}\frac{\sin x^2(1+\cos x)}{1-\cos^2 x} = \lim_{x\to 0^-}\frac{\sin\left(x^2\right)(1+\cos x)}{\sin^2 x} =$$

$$= \lim_{x\to 0^-}\left(\frac{\sin x^2}{x^2}\times\frac{x^2}{\sin^2 x}\right)\times\lim_{x\to 0}(1+\cos x) =$$

$$= \lim_{x\to 0}\frac{\sin x^2}{x^2}\times\frac{1}{\left(\lim\limits_{x\to 0^-}\frac{\sin x}{x}\right)^2}\times(1+1) = \qquad \left| \begin{array}{l} y = x^2 \\ \text{Se } x \to 0^-, y \to 0^+ \end{array}\right.$$

$$= \lim_{y\to 0^+}\frac{\sin y}{y}\times\frac{1}{1^2}\times 2 = 1\times 1\times 2 = 2$$

- $f(0) = 2$

Como $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$, existe $\lim\limits_{x\to 0} f(x)$ pelo que f é contínua no ponto $x = 0$.

29.2. Assíntotas verticais

f é contínua em $\left]-2\pi, +\infty\right[$, pois é contínua em $\left]-2\pi, 0\right[$ por ser definida pela composta, diferença e quociente entre duas funções contínuas (função seno, função cosseno e funções polinomiais)e em $\left]0, +\infty\right[$ por ser definida pelo quociente entre de funções contínuas (funções polinomiais) bem como no ponto $x = 0$. Logo apenas a reta de equação $x = -2\pi$ poderá ser assíntota vertical do seu gráfico.

$$\lim_{x\to 2-\pi^+} f(x) = \lim_{x\to -2\pi^+}\frac{\sin x^2}{1-\cos x} = \frac{\sin\left(4\pi^2\right)}{0^+} = +\infty$$

Portanto, a reta de equação $x = -2\pi$ é assíntota ao gráfico de f .

Assíntotas não verticais $(y = mx+b)$:

$$m = \lim_{x\to +\infty}\frac{f(x)}{x} = \lim_{x\to +\infty}\frac{x^2+2x}{x^2} = \lim_{x\to +\infty}\frac{x^2}{x^2} = 1$$

$$b = \lim_{x\to +\infty}(f(x)-mx) = \lim_{x\to +\infty}\left(\frac{x^2+2x}{x}-x\right) =$$

$$= \lim_{x\to +\infty}\frac{x^2+2x-x^2}{x} = \lim_{x\to +\infty}\frac{2x}{x} = 2$$

A reta de equação $x = x+2$ é assíntota ao gráfico de f em $+\infty$.

29.3. A função f é contínua no intervalo $\left[-6, -\frac{3\pi}{2}\right]$, pois $\left[-6, -\frac{3\pi}{2}\right] \subset D_f$ e como vimos na alínea anterior, f é contínua.

- $f(-6) = \frac{\sin 36}{1-\cos(-6)} \approx -24{,}900$
- $f\left(-\frac{3\pi}{2}\right) = \frac{\sin\left(-\frac{3\pi}{2}\right)^2}{1-\cos\left(-\frac{3\pi}{2}\right)} \approx -0{,}214$

Como f é contínua no intervalo $\left[-6, -\frac{3\pi}{2}\right]$ e $f(-6) < -4 < f\left(-\frac{3\pi}{2}\right)$, podemos então concluir, pelo Teorema de Bolzano que $\exists c \in \left]-6, -\frac{3\pi}{2}\right[: f(c) = -4$.

30.1. $\lim\limits_{x\to\frac{3\pi}{2}} \left[(1+\sin x)\times \tan^2 x\right] \overset{(0\times\infty)}{=} \lim\limits_{x\to\frac{3\pi}{2}} \left[(1+\sin x)\times \frac{\sin^2 x}{\cos x}\right] =$

$= \lim\limits_{x\to\frac{3\pi}{2}} \frac{1+\sin x}{\cos^2 x} \times \lim\limits_{x\to\frac{3\pi}{2}} \sin^2 x =$

$= \lim\limits_{y\to 0} \frac{1+\sin\left(\frac{3\pi}{2}+y\right)}{\left[\cos\left(\frac{3\pi}{2}+y\right)\right]^2} \times \sin^2 \frac{3\pi}{2} =$

$y = x - \frac{3\pi}{2} \Leftrightarrow x = \frac{3\pi}{2} + y$

Se $x \to \frac{3\pi}{2} \Rightarrow y \to 0^+$

$= \lim\limits_{y\to 0} \frac{1-\cos y}{(\sin y)^2} \times (-1)^2 = \lim\limits_{y\to 0} \frac{(1-\cos y)(1+\cos y)}{\sin^2 y(1+\cos y)} \times 1 =$

$= \lim\limits_{y\to 0} \frac{1-\cos^2 y}{\sin^2 y} \times \lim\limits_{y\to 0} \frac{1}{1+\cos y} = \lim\limits_{y\to 0} \frac{\sin^2 y}{\sin^2 y} \times \frac{1}{1+1} = \frac{1}{2}$

30.2. $\lim\limits_{x\to 0} \frac{2\sin x - \sin(2x)}{x^3 \cos x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0} \frac{2\sin x - 2\sin x\cos x}{x^3 \cos x} =$

$= \lim\limits_{x\to 0} \frac{2\sin x(1-\cos x)}{x^3 \cos x} = \lim\limits_{x\to 0} \frac{2\sin x(1-\cos x)(1+\cos x)}{x^2 \cos x(1+\cos x)} =$

$= \lim\limits_{x\to 0} \frac{2\sin x\left(1-\cos^2 x\right)}{x^3 \cos x(1+\cos x)} = \lim\limits_{x\to 0} \frac{2\sin x\left(\sin^2 x\right)}{x^3 \cos x(1+\cos x)} =$

$= \lim\limits_{x\to 0} \frac{2\sin^3 x}{x^3 (1+\cos x)\cos x} = 2\left(\lim\limits_{x\to 0} \frac{\sin x}{x}\right)^3 \lim\limits_{x\to 0} \frac{1}{(1+\cos x)\cos x} =$

$= 2\times 1^3 \times \frac{1}{(1+1)\times 1} = 1$

31.

- $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} \frac{\sin(-3x)}{-2x} \overset{\left(\frac{0}{0}\right)}{=}$

  $y = -3x$

  Se $x \to 0^-, y \to 0^+$

  $= \lim\limits_{x\to 0^-} \left[\frac{\sin(-3x)}{-3x} \times \frac{-3x}{-2x}\right] = \frac{3}{2} \lim\limits_{y\to 0^+} \frac{\sin y}{y} = \frac{3}{2} \times 1 = \frac{3}{2}$

- $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} \frac{2x - \tan\frac{x}{2}}{x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0^+} \left[2 - \frac{\tan\frac{x}{2}}{x}\right] =$

  $= 2 - \lim\limits_{x\to 0^+} \frac{\tan\frac{x}{2}}{x} = 2 - \lim\limits_{x\to 0^+} \frac{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{x} =$

  $= 2 - \lim\limits_{x\to 0^+} \frac{\sin\frac{x}{2}}{x\cos\frac{x}{2}} = 2 - \lim\limits_{x\to 0^+} \frac{\sin\frac{x}{2}}{2\times\frac{x}{2}} \times \lim\limits_{x\to 0^+} \frac{1}{\cos\frac{x}{2}} =$

  $= 2 - \frac{1}{2} \lim\limits_{x\to 0^+} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \times \frac{1}{1} =$

  $y = \frac{x}{2}$

  Se $x \to 0^+ \Rightarrow y \to 0^+$

  $= 2 - \frac{1}{2} \lim\limits_{y 0^+} \frac{\sin y}{y} \times 1 = 2 - \frac{1}{2} \times 1 \times \frac{1}{1} = 2 - \frac{1}{2} = \frac{3}{2}$

- $g(0) = \frac{3}{2}$

Como $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^-} g(x) = g(0)$, podemos então concluir que existe $\lim\limits_{x\to 0} g(x)$ pelo que a função g é contínua no ponto $x = 0$.

Ficha para praticar 12

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1.1. $f'(x) = \left(3\cos\frac{x}{2}\right)' = 3\left(\cos\frac{x}{2}\right)' = 3\left(-\frac{x'}{2}\sin\frac{x}{2}\right) = -\frac{3}{2}\sin\frac{x}{2}$

1.2. $g'(x) = \left(2x + \sin\frac{\pi}{x}\right)' = (2x)' + \left(\sin\frac{\pi}{x}\right)' =$

$= 2 + \frac{\pi'}{x}\cos\frac{\pi}{x} = 2 + \left(-\frac{\pi}{x^2}\right)\cos\frac{\pi}{x} = 2 - \frac{\pi}{x^2}\cos\frac{\pi}{x}$

1.3. $h'(x) = (\sin x\cos x)' =$

$= (\sin x)'\cos x + \sin x(\cos x)' =$

$= \cos x\cos x + \sin x(-\sin x) =$

$= \cos^2 x - \sin^2 x = \cos(2x)$

1.4. $j'(x) = \left(4\sin\left(\pi x^2\right)\right)' = 4\left(\sin\left(\pi x^2\right)\right)' =$

$= 4\left(\pi x^2\right)'\cos\left(\pi x^2\right) = 4(2\pi x)\cos\left(\pi x^2\right) =$

$= 8\pi x\cos\left(\pi x^2\right)$

1.5. $p'(x) = \left(\sin^2 x\right)' = \left[(\sin x)^2\right]' =$

$= 2\sin x(\sin x)' = 2\sin x\cos x = \sin(2x)$

1.6. $r'(x) = \left(\frac{1+\sin x+\cos x}{\cos x}\right)' =$

$= \frac{(1+\sin x+\cos x)'\cos x - (1+\sin x+\cos x)\cdot(\cos x)'}{(\cos x)^2} =$

$= \frac{(\cos x-\sin x)\cos x - (1+\sin x+\cos x)\cdot(-\sin x)}{\cos^2 x} =$

$= \frac{\cos^2 x - \sin x\cos x + \sin x + \sin^2 x + \sin x\cos x}{\cos^2 x} =$

$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2(x)} =$

$= \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{1-\sin^2 x} = \frac{1+\sin x}{(1+\sin x)(1-\sin x)} = \frac{1}{1-\sin x}$

1.7. $s'(x) = \left(2 + \tan x - \frac{1}{\cos(2x)}\right)' =$

$= (2)' + (\tan x)' - \left(\frac{1}{\cos(2x)}\right)' =$

$= 0 + \frac{1}{\cos^2(x)} - \frac{-1\times(\cos(2x))'}{(\cos(2x))^2} =$

$= \frac{1}{\cos^2 x} - \frac{-(-2\sin(2x))}{\cos^2(x)} = \frac{1}{\cos^2 x} - \frac{2\sin(2x)}{\cos^2(2x)}$

1.8. $t'(x)=\left(\tan(x)\cos^2(x)\right)'=$

$=(\tan x)'\cos^2 x+\tan x\left(\cos^2 x\right)'=$

$=\frac{1}{\cos^2 x}\cos^2 x+\tan x\left(2\cos x(-\sin x)\right)=$

$=1-\tan x(2\sin x\cos x)=$

$=1-\tan x-\sin(2x)$ ou $1-\tan x\sin(2x)=$

$=1-\frac{\sin x}{\cos x}(2\sin x\cos x)=1-2\sin^2(x)$

1.9. $v'(x)=\left(\sqrt{8-4\cos(\pi x)}\right)'=\frac{\left(8-4\cos(\pi x)\right)'}{2\sqrt{8-4\cos(\pi x)}}=$

$=\frac{0-4\left(\cos(\pi x)\right)'}{2\sqrt{8-4\cos(\pi x)}}=\frac{-4\left(-\pi\sin(\pi x)\right)}{2\sqrt{8-4\cos(\pi x)}}=$

$=\frac{4\pi\sin(\pi x)}{2\sqrt{8-4\cos(\pi x)}}=\frac{2\pi\sin(\pi x)}{\sqrt{8-4\cos(\pi x)}}$

1.10. $w'(x)=\left(\tan x+\frac{1}{\tan x}+\sin x\right)'=$

$=\frac{1}{\cos^2(x)}+\frac{-1(\tan x)'}{(\tan x)^2}+\cos x=$

$=\frac{1}{\cos^2(x)}+\frac{-\frac{1}{\cos^2(x)}}{\tan^2(x)}+\cos(x)=$

$=\frac{1}{\cos^2(x)}-\frac{\frac{1}{\cos^2(x)}}{\frac{\sin^2(x)}{\cos^2(x)}}+\cos(x)=\frac{1}{\cos^2 x}-\frac{1}{\sin^2 x}+\cos x$

1.11. $i'(x)=\left(\sqrt{\frac{2-\cos x}{2-\sin(3x)}}\right)'=\frac{\left(\frac{2-\cos x}{2-\sin(3x)}\right)'}{2\sqrt{\frac{2-\cos x}{2-\sin(3x)}}}=$

$=\frac{\frac{(2-\cos x)'(2-\sin(3x))-(2-\cos x)(2-\sin(3x))'}{(2-\sin(3x))^2}}{2\sqrt{\frac{2-\cos(x)}{2-\sin(3x)}}}=$

$=\frac{\frac{\sin x(2-\sin(3x))-(2-\cos x)(-3\cos(3x))}{(2-\sin(3x))^2}}{2\sqrt{\frac{2-\cos(x)}{2-\sin(3x)}}}=$

$=\frac{2\sin x-\sin x\sin(3x)+6\cos(3x)-3\cos x\cos(3x)}{2(2-\sin(3x))^2\sqrt{\frac{2-\cos(x)}{2-\sin(3x)}}}$

1.12. $l'(x)=\left(\cos(\sin x)-x^2\right)'=-(\sin x)'\sin(\sin(x))-2x=$

$=-\cos x\sin(\sin x)-2x$

2.1. $f'(x)=\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)'=$

$=\frac{(\sin x+\cos x)'(\sin x-\cos x)-(\sin x+\cos x)(\sin x-\cos x)'}{(\sin x-\cos x)^2}=$

$=\frac{(\cos x-\sin x)(\sin x-\cos x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}=$

$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}=$

Cálculo auxiliar

$(\sin x-\cos x)^2=\sin^2 x-2\sin x\cos x+\cos^2 x=$

$=\sin^2 x+\cos^2 x-\sin(2x)=1-\sin(2x)$

$(\sin x+\cos x)^2=\sin^2 x+2\sin x\cos x+\cos^2 x=$

$=1+\sin(2x)$

$f'(x)=\frac{-(1-\sin(2x))-(1+\sin(2x))}{1-\sin(2x)}=$

$=\frac{-2}{1-\sin(2x)}=\frac{2}{\sin(2x)-1}$

$=\frac{-2}{1-\sin(2x)}=\frac{2}{\sin(2x)-1}$

2.2. $g'(x)=\left(\sin^3(2x)\right)'=3\sin^2(2x)\left(\sin(2x)\right)'=$

$=3\sin^2(2x)\times 2\cos(2x)=$

$=3\sin(2x)\times 2\sin(2x)\cos(2x)=$

$=3\sin(2x)\sin(4x)$

2.3. $h'(x)=\left(\sin^2 x+\sin x\right)'=\left(\sin^2 x\right)'+(\sin x)'=$

$=2\sin x(\sin x)'+\cos x=2\sin x\cos x+\cos x=$

$=\sin(2x)+\cos x$

2.4. $j(x)=\left(\tan^2\left(x^4\right)+\frac{1}{\tan(x)}\right)'=$

$=2\tan\left(x^4\right)\left(\tan\left(x^4\right)\right)'-\frac{-1\times(\tan(x))'}{(\tan(x))^2}=$

$=2\tan\left(x^4\right)\times\frac{\left(x^4\right)'}{\cos^2\left(x^4\right)}+\frac{\frac{1}{\cos^2(x)}}{\tan^2(x)}=$

$=2\tan\left(x^4\right)\times\frac{4x^3}{\cos^2\left(x^4\right)}+\frac{\frac{1}{\cos^2(x)}}{\frac{\sin^2(x)}{\cos^2(x)}}=$

$=2\frac{\sin\left(x^4\right)}{\cos\left(x^4\right)}\times\frac{4x^3}{\cos^2\left(x^4\right)}+\frac{1}{\sin^2 x}=\frac{8x^3\sin\left(x^4\right)}{\cos^3\left(x^4\right)}+\frac{1}{\sin^2(x)}$

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3.1. $f'(x)=\left(\sin^2 x+2\cos^3 x\right)'=$

$=2\sin x(\sin x)'+2\times 3\cos^2 x(\cos x)'=$

$=2\sin x\cos x+6\cos^2 x(-\sin x)=$

$=2\sin x\cos x-6\cos^2 x\sin x$

$$f'\left(\frac{\pi}{2}\right)=2\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)-6\cos^2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)=$$
$$=2\times1\times0-6\times0^2\times1=0$$

3.2. $g'(x)=\left(x\sin x+\cos\left(\frac{x}{2}\right)\right)'=$

$$=(x)'\sin x+x(\sin x)'+\left(\frac{x}{2}\right)'\left(-\sin\frac{x}{2}\right)=$$
$$=\sin x+x(-\cos x)+\frac{1}{2}\left(-\sin\frac{x}{2}\right)=$$
$$=\sin x-x\cos x-\frac{1}{2}\sin\frac{x}{2}$$
$$g'\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)-\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin\left(\frac{\pi}{4}\right)=$$
$$=1-\frac{\pi}{2}\times0-\frac{1}{2}\times\frac{\sqrt{2}}{2}=1-\frac{\sqrt{2}}{4}$$

3.3. $h'(x)=\left(\sin\left(x^2\right)-\sin^2 x\right)'=$

$$=\left(x^2\right)'\cos\left(x^2\right)-2\sin x(\sin x)'=$$
$$=2x\cos\left(x^2\right)-2\sin x\cos x$$
$$h'\left(\frac{\pi}{2}\right)=2\times\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)^2-2\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)=$$
$$=\pi\cos\frac{\pi^2}{4}-2\times1\times0=\pi\cos\left(\frac{\pi^2}{4}\right)$$

3.4. $j'(x)=\left(\cos\left(x^3\right)-x\cos(2x)\right)'=$

$$=\left(-x^3\right)'\sin\left(x^3\right)-\left[(x)'\cos(2x)+x\left(\cos(2x)\right)'\right]=$$
$$=-3x^2\sin\left(x^3\right)-\left[\cos(2x)+x\left(-2\sin(2x)\right)\right]=$$
$$=-3x^2\sin\left(x^3\right)-\cos(2x)-2x\sin(2x)$$
$$j'\left(\frac{\pi}{2}\right)=-3\left(\frac{\pi}{2}\right)^2\sin\left(\frac{\pi}{2}\right)^3-\cos\left(2\times\frac{\pi}{2}\right)-2\times\frac{\pi}{2}\sin\left(2\times\frac{\pi}{2}\right)=$$
$$=-3\left(\frac{\pi^2}{4}\right)\sin\left(\frac{\pi^3}{8}\right)-\cos\pi-\pi\sin\pi=$$
$$=-\frac{3\pi^2}{4}\sin\left(\frac{\pi^3}{8}\right)-(-1)-\pi\times0=-\frac{3\pi^2}{4}\sin\left(\frac{\pi^3}{8}\right)+1$$

4.1. $f'(0)=\lim\limits_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0}\frac{\sin(2x)-\sin(2\times0)}{x-0}=$

$$=\lim_{x\to0}\frac{\sin(2x)}{x}=2\times\lim_{x\to0}\frac{\sin(2x)}{2x}=$$ | $y=2x$; Se $x\to0, y\to0$
$$=2\lim_{y\to0}\frac{\sin y}{y}=2\times1=2$$

Portanto, $f'(0)=2$

4.2. $f'\left(\frac{\pi}{2}\right)=\lim\limits_{h\to0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}=\lim\limits_{h\to0}\frac{\cos\left(\frac{\pi}{2}+h\right)-\cos\frac{\pi}{2}}{h}=$

$$=\lim_{h\to0}\frac{-\sin h-0}{h}=-\lim_{h\to0}\frac{\sin h}{h}=-1$$

Portanto $f'\left(\frac{\pi}{2}\right)=-1$

4.3. $f'(\pi)=\lim\limits_{h\to0}\frac{f(\pi+h)-\tan\pi}{h}=$

$$=\lim_{h\to0}\frac{\tan(\pi+h)-\tan\pi}{h}=$$
$$=\lim_{h\to0}\frac{\tan h-0}{h}=$$
$$=\lim_{h\to0}\frac{\frac{\sin h}{\cos h}}{h}=\lim_{h\to0}\frac{\sin h}{h}\times\lim_{h\to0}\frac{1}{\cos h}=1\times\frac{1}{1}=1$$

Portanto, $f'(\pi)=1$

4.4. $f'(0)=\lim\limits_{x\to0}\frac{f(x)-f(0)}{x-0}=$

$$=\lim_{x\to0}\frac{2\sin(x)-\sin(2x)-0}{x}=$$
$$=2\lim_{x\to0}\frac{\sin x}{x}-\lim_{x\to0}\frac{\sin(2x)}{x}=$$
$$=2\times1-2\lim_{x\to0}\frac{\sin(2x)}{2x}=$$ | $y=2x$; Se $x\to0, y\to0$
$$=2-2\lim_{y\to0}\frac{\sin(y)}{y}=2-2\times1=2-2=0$$

Portanto, $f'(0)=0$.

5.1.
- $f'(x)=\left(2\sin\frac{\pi x}{3}\right)'=2\left(\frac{\pi x}{3}\right)'\cos\frac{\pi x}{3}=$
  $$=\frac{2\pi}{3}\cos\left(\frac{\pi x}{3}\right)$$
- $g'(x)=\left[(\sin x+\cos x)^2\right]'=$
  $$=2(\sin x+\cos x)(\sin x+\cos x)'=$$
  $$=2(\sin x+\cos x)(\cos x-\sin x)=$$
  $$=2\left(\cos^2x-\sin^2x\right)=2\cos(2x)$$
- $h'(x)=(x-\cos(2x))'=x'-(\cos(2x))'=$
  $$=1-(-2\sin(2x))=1+2\sin(2x)$$

Portanto, $f'(x)=\frac{2\pi}{3}\cos\left(\frac{\pi x}{3}\right)$, $g'(x)=2\cos(2x)$ e $h'(x)=1+2\sin(2x)$

5.2. Zeros de f':

$$f'(x)=0\Leftrightarrow\frac{2\pi}{3}\cos\left(\frac{\pi x}{3}\right)=0\Leftrightarrow\cos\left(\frac{\pi x}{3}\right)=0\Leftrightarrow$$
$$\Leftrightarrow\frac{\pi x}{3}=\frac{\pi}{2}+k\pi,k\in\mathbb{Z}\Leftrightarrow$$
$$\Leftrightarrow\pi x=\frac{3\pi}{2}+3k\pi,\ k\in\mathbb{Z}\Leftrightarrow x=\frac{3}{2}+3k,\ k\in\mathbb{Z}$$

Zeros de g':

$$g'(x)=0\Leftrightarrow2\cos(2x)=0\Leftrightarrow\cos(2x)=0\Leftrightarrow$$
$$\Leftrightarrow2x=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\Leftrightarrow x=\frac{\pi}{4}+\frac{k\pi}{2},\ k\in\mathbb{Z}$$

Zeros de h':

$h'(x)=0 \Leftrightarrow 1+2\sin(2x)=0 \Leftrightarrow \sin(2x)=-\frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \sin(2x)=\sin\left(-\frac{\pi}{6}\right) \Leftrightarrow$

$\Leftrightarrow 2x=-\frac{\pi}{6}+2k\pi \vee 2x=\frac{7\pi}{6}+2k\pi,\ k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x=-\frac{\pi}{12}+k\pi \vee x=\frac{7\pi}{12}+k\pi,\ k\in\mathbb{Z}$

6.1. $f(x)=2x+\sin\frac{\pi}{x}$

$f'(x)=\left(2x+\sin\frac{\pi}{x}\right)'=(2x)'+\left(\sin\frac{\pi}{x}\right)'=$

$=2+\left(\frac{\pi}{x}\right)'\cos\frac{\pi}{x}=2-\frac{\pi}{x^2}\cos\frac{\pi}{x}$

Uma equação da reta tangente ao gráfico de f no ponto de abcissa $x=-1$ é:

$y-f(-1)=f'(-1)(x-(-1))$

$f(-1)=-2+\sin(-\pi)=-2+0=-2$

$f'(-1)=2-\frac{\pi}{(-1)^2}\cos\left(\frac{\pi}{-1}\right)=2-\pi\cos(-\pi)=2+\pi$

$y+2=(2+\pi)(x+1) \Leftrightarrow y=2x+2+\pi x+\pi-2$

$\Leftrightarrow y=2x+\pi x+\pi \Leftrightarrow y=(2+\pi)x+\pi$

Portanto, $y=(2+\pi)x+\pi$ é uma equação da reta pedida.

6.2. $f(x)=3+\tan x-\tan^2 x$

$f'(x)=\left(3+\tan x-\tan^2 x\right)'=(3)'+(\tan x)'-\left(\tan^2 x\right)'=$

$=0+\frac{1}{\cos^2 x}-2\tan x(\tan x)'=\frac{1}{\cos^2 x}-2\tan x\times\frac{1}{\cos^2 x}$

$f(\pi)=3+\tan(\pi)-\tan^2(\pi)=3+0-0^2=3$

$f'(\pi)=\frac{1}{\cos^2\pi}-2\tan\pi\times\frac{1}{\cos^2\pi}=\frac{1}{1}-2\times 0\times\frac{1}{1}=1$

$y-3=1(x-\pi) \Leftrightarrow y=x-\pi+3$

Portanto, $y=x-\pi+3$ é uma equação da reta pedida.

6.3. $f(x)=2\cos^2\frac{x}{2}$

$f'(x)=2\left(\cos^2\frac{x}{2}\right)'=2\times 2\cos\frac{x}{2}\left(\cos\frac{x}{2}\right)'=$

$=4\cos\frac{x}{2}\left(-\left(\frac{x}{2}\right)'\sin\frac{x}{2}\right)=4\cos\frac{x}{2}\left(-\frac{1}{2}\sin\frac{x}{2}\right)=$

$=-2\cos\frac{x}{2}\sin\frac{x}{2}=-\sin(x)$

Uma equação de reta tangente ao gráfico de f no ponto de abcissa $x=\frac{\pi}{2}$ é:

$y-f\left(\frac{\pi}{2}\right)=f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)$

$f\left(\frac{\pi}{2}\right)=2\cos^2\left(\frac{\pi}{4}\right)=2\left(\cos\left(\frac{\pi}{4}\right)\right)^2=2\left(\frac{\sqrt{2}}{2}\right)^2=1$

$f'\left(\frac{\pi}{2}\right)=-\sin\left(\frac{\pi}{2}\right)=-1$

Assim, $y-1=-1\left(x-\frac{\pi}{2}\right) \Leftrightarrow y=-x+\frac{\pi}{2}+1$.

Portanto, $y=-x+\frac{\pi}{2}+1$ é uma equação da reta pedida.

6.4. $f(x)\frac{\sin x}{1-2\sin x}$

$f'(x)=\left(\frac{\sin x}{1-2\sin x}\right)'=\frac{(\sin x)'(1-2\sin x)-\sin x(1-2\sin x)'}{(1-2\sin x)^2}=$

$=\frac{\cos x(1-2\sin x)-\sin x(-2\cos x)}{(1-2\sin x)^2}=$

$=\frac{\cos x-2\sin x\cos x+2\sin x\cos x}{(1-2\sin x)^2}=\frac{\cos(x)}{(1-2\sin x)^2}$

Uma equação da reta tangente ao gráfico de f no ponto de abcissa $x=-\pi$ é:

$y-f(-\pi)=f'(-\pi)(x-(-\pi))$

$f(-\pi)=\frac{\sin(-\pi)}{1-2\sin(-\pi)}=\frac{0}{1-2\times 0}=\frac{0}{1}=0$

$f'(-\pi)=\frac{\cos(-\pi)}{(1-2\sin(-\pi))^2}=\frac{-1}{(1-2\times 0)^2}=\frac{-1}{1}=-1$

Assim, $y-0=-1(x-(-\pi)) \Leftrightarrow y=-x-\pi$

Portanto, $y=-x-\pi$ é uma equação da reta pedida.

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7.1. $f(x)=\sin(2x)$

$f'(x)=(\sin(2x))'=(2x)'\cos(2x)=2\cos(2x)$

$f''(x)=(2\cos(2x))'=2(\cos(2x))'=2\left[-(2x)'\sin(2x)\right]=$

$=-4\sin(2x)$

Logo, $f''(x)=-4\sin(2x)$.

7.2. $f(x)=\cos(2x)-x^2$

$f'(x)=\left(\cos(2x)-x^2\right)'=(\cos(2x))'-\left(x^2\right)'=$

$=-(2x)'\sin(2x)-2x=-2\sin(2x)-2x$

$f''(x)=(-2\sin(2x)-2x)'=(-2\sin(2x))'-(2x)'=$

$=-2(\sin(2x))'-2=$

$=-2\left((2x)'\cos(2x)\right)-2=-4\cos(2x)-2$

Portanto, $f''(x)=-4\cos(2x)-2$

7.3. $f(x)=\tan(2x)$

$f'(x)=(\tan(2x))'=\frac{(2x)'}{\cos^2(2x)}=\frac{2}{\cos^2(2x)}$

$$f''(x)=\left(\frac{2}{\cos^2(2x)}\right)'=\frac{(2)'\cos^2(2x)-2\left(\cos^2(2x)\right)'}{\left(\cos^2(2x)\right)^2}=$$

$$=\frac{0-2\times 2\cos(2x)\left(\cos(2x)\right)'}{\cos^4(2x)}=$$

$$=\frac{-4\cos(2x)\left(-(2x)'\sin(2x)\right)}{\cos^4(2x)}=\frac{8\cos(2x)\sin(2x)}{\cos^4(2x)}=$$

$$=\frac{8\sin(2x)}{\cos^3(2x)}$$

Portanto, $f''(x)=\dfrac{8\sin(2x)}{\cos^3(2x)}$.

7.4. $f(x)=\dfrac{\cos x}{1+\sin x}$

$$f'(x)=\left(\frac{\cos x}{1+\sin x}\right)'=\frac{(\cos x)'(1+\sin x)-\cos x(1+\sin x)'}{(1+\sin x)^2}=$$

$$=\frac{-\sin x(1+\sin x)-\cos x\cos x-\sin x(1+\sin x)-\cos x\cos x}{(1+\sin x)^2}=$$

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}=$$

$$=\frac{-\sin(x)-1}{(1+\sin x)^2}=-\frac{1+\sin x}{(1+\sin x)^2}=-\frac{1}{1+\sin x}=$$

$$f''(x)=\left(-\frac{1}{1+\sin x}\right)'=-\frac{(1+\sin x)'}{(1+\sin x)^2}=\frac{\cos x}{(1+\sin x)^2}$$

Portanto, $f''(x)=\dfrac{\cos x}{(1+\sin x)^2}$

8.1. Para $x>0$, tem-se:

$$f'(x)=(x+\sin x)'=(x)'+(\sin x)'=1+\cos x$$

Para $x<0$, tem-se:

$$f'(x)=\left(\frac{16-4x}{x^2-6x+8}\right)'=$$

$$=\frac{(16-4x)'\left(x^2-6x+8\right)-(16-4x)\left(x^2-6x+8\right)'}{\left(x^2-6x+8\right)^2}=$$

$$=\frac{-4\left(x^2-6x+8\right)-(16-4x)(2x-6)}{\left(x^2-6x+8\right)^2}=$$

$$=\frac{-4x^2+24x-32-32x+96+8x^2-24x}{\left(x^2-6x+8\right)^2}=$$

$$=\frac{4x^2-32x+64}{\left(x^2-6x+8\right)^2}==\frac{4(x-4)^2}{\left[(x-2)(x-4)\right]^2}=\frac{4}{(x-2)^2}$$

Para $x=0$, tem-se:

$$f'\left(0^+\right)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=$$

$$=\lim_{x\to0^+}\frac{x+\sin(x)-\left(0+\sin(0)\right)}{x}=$$

$$=\lim_{x\to0^+}\frac{x+\sin(x)}{x}=\lim_{x\to0^+}\frac{x}{x}+\lim_{x\to0^+}\frac{\sin(x)}{x}=2$$

$$f'\left(0^-\right)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{\frac{16-4x}{x^2-6x+8}-0}{x}=$$

$$=\lim_{x\to0^-}\frac{\frac{-4(x-4)}{(x-2)(x-4)}}{x}=\lim_{x\to0^-}\frac{\frac{-4}{x-2}}{x}=\lim_{x\to0^-}\frac{-4}{x^2-2x}=-\infty$$

Logo, não existe $f'(0)$.

A função f' pode ser caracterizada do modo que se segue:

$f':\mathbb{R}\setminus\{0\}\to\mathbb{R}$

$$x\mapsto\begin{cases}1+\cos(x) & \text{se } x>0\\ \dfrac{4}{(x-2)^2} & \text{se } x<0\end{cases}$$

8.2. Para $x<0$ **:** $g'(x)=(\cos x)'=-\sin x$

Para $x>0$: $g'(x)=\left(\sin(2x)\right)'=2\cos(2x)$

Para $x=0$, tem-se:

$$g'\left(0^-\right)=\lim_{x\to0^-}\frac{g(x)-g(0)}{x-0}=\lim_{x\to0^-}\frac{\cos x-\cos(0)}{x}$$

$$=\lim_{x\to0^-}\frac{\cos(x)-1}{x}=\lim_{x\to0^-}\frac{(\cos x-1)(\cos x+1)}{x(\cos x+1)}=$$

$$=\lim_{x\to0^-}\frac{\cos^2 x-1}{x(\cos x+1)}=\lim_{x\to0^-}\frac{-\sin^2 x}{x(\cos x+1)}=$$

$$=\lim_{x\to0^-}\frac{\sin x}{x}\times\lim_{x\to0^-}\frac{-\sin x}{\cos x+1}=1\times\frac{0}{1+1}=0$$

$$g'\left(0^+\right)=\lim_{x\to0^+}\frac{g(x)-g(0)}{x-0}=\lim_{x\to0^+}\frac{\sin(2x)-\cos(0)}{x}$$

$$=\lim_{x\to0^+}\frac{\sin(2x)-1}{x}=\frac{0-1}{0^+}=-\infty$$

Como $g'\left(0^+\right)=-\infty$ não existe $g'(0)$.

Assim, uma caracterização da função g' é:

$g':\mathbb{R}\setminus\{0\}\to\mathbb{R}$

$$x\mapsto\begin{cases}-\sin(x) & \text{se } x<0\\ 2\cos(2x) & \text{se } x>0\end{cases}$$

9.1. $f(x)=\begin{cases}\dfrac{\sin x}{|x|} & \text{se } x\neq0\\ -1 & \text{se } x=0\end{cases}\Leftrightarrow f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{se } x>0\\ \dfrac{\sin x}{-x} & \text{se } x<0\\ -1 & \text{se } x=0\end{cases}$

$$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{\sin x}{x}=1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{-x} = -\lim_{x\to 0^-} \frac{\sin x}{x} = -1$$

Como $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ não existe $\lim_{x\to 0} f(x)$ pelo que a função f não é contínua em $x = 0$.

9.2. Para $x > 0$, tem-se:

$$f'(x) = \left(\frac{\sin x}{x}\right)' = \frac{(\sin x)' x - x' \sin x}{x^2} = \frac{x\cos x}{x^2}$$

Para $x < 0$, tem-se:

$$f'(x) = \left(\frac{\sin x}{-x}\right)' = \left(\frac{\sin x}{x}\right)' = -\frac{x\cos x - \sin x}{x^2} = \frac{x\cos x}{x^2}$$

Por outro lado, como f não é contínua em $x = 0$ então não é diferenciável em $x = 0$, ou seja, não existe $f'(0)$.

Assim, a função f pode caracterizar-se do modo seguinte:

$$f' : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

$$x \mapsto \begin{cases} \dfrac{x\cos x - \sin x}{x^2} & \text{se } x > 0 \\ \dfrac{\sin x - x\cos x}{x^2} & \text{se } x < 0 \end{cases}$$

10.1. Para $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, tem-se:

$$g'(x) = \left(2\sin^2(2x)\right)' = 2\left[\left(\sin(2x)\right)^2\right]' =$$

$$= 2 \times 2\sin(2x)\left(\sin(2x)\right)' = 2\times\left(2\sin(2x)\right)\left(2\cos(2x)\right) =$$

$$= 4 \times 2\sin(2x)\cos(2x) = 4\sin(4x)$$

$$g'(x) = 0 \Leftrightarrow 4\sin(4x) = 0 \Leftrightarrow \sin(4x) = 0 \Leftrightarrow$$

$$\Leftrightarrow 4x = k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{k\pi}{4}, k \in \mathbb{Z}$$

Como $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

$$g'(x) = 0 \Leftrightarrow x = -\frac{\pi}{2} \vee x = -\frac{\pi}{4} \vee x = 0 \vee x = \frac{\pi}{4} \vee x = \frac{\pi}{2}$$

x	$-\frac{\pi}{2}$		$-\frac{\pi}{4}$		0		$\frac{\pi}{4}$		$\frac{\pi}{2}$
g'	0	+	0	–	0	+	0	–	0
g	0	↗	2	↘	0	↗	2	↘	0
	Mín		Máx		Mín		Máx		Mín

A função g é estritamente decrescente em $\left[-\frac{\pi}{4}, 0\right]$ e em $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ e é estritamente crescente em $\left[-\frac{\pi}{2}, -\frac{\pi}{4}\right]$ e em $\left[0, \frac{\pi}{4}\right]$. A função g tem mínimos relativos iguais a 0 para $x = -\frac{\pi}{2}$, $x = 0$ e $x = \frac{\pi}{2}$ e máximos relativos iguais a 2 para $x = -\frac{\pi}{4}$ e $x = \frac{\pi}{4}$.

10.2. As abcissas pedidas são as soluções da equação $g'(x) = -4$.

Assim:

$$g'(x) = -4 \Leftrightarrow 4\sin(4x) = -4 \Leftrightarrow \sin(4x) = -1 \Leftrightarrow$$

$$\Leftrightarrow 4x = \frac{-\pi}{2} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow x = -\frac{\pi}{8} + \frac{k\pi}{2}, k \in \mathbb{Z}$$

As abcissas dos pontos onde as retas tangentes ao gráfico de g têm declive -4 são, portanto, $x = -\frac{\pi}{8} + \frac{k\pi}{2}, k \in \mathbb{Z}$.

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11.1. Para $x \in \left]-\pi, \frac{\pi}{2}\right[$, tem-se:

$$h'(x) = \left(\frac{\cos x}{1+\cos x}\right)' = \frac{(\cos x)'(1+\cos x) - \cos x(1+\cos x)'}{(1+\cos x)^2} =$$

$$= \frac{-\sin x(1+\cos x) - \cos x(-\sin x)}{(1+\cos x)^2} =$$

$$= \frac{-\sin x - \sin x\cos x + \cos x\sin x}{(1+\cos x)^2} = \frac{-\sin(x)}{(1+\cos x)^2}$$

$$h'(x) = 0 \Leftrightarrow \frac{-\sin(x)}{\left(1+\cos(x)\right)^2} = 0 \Leftrightarrow$$

$$\Leftrightarrow -\sin(x) = 0 \wedge x \in D_h \Leftrightarrow$$

$$\Leftrightarrow \sin(x) = 0 \wedge x \in \left]-\pi, \frac{\pi}{2}\right[\Leftrightarrow x = 0$$

x	$-\pi$		0		$\frac{\pi}{2}$
h'		+	0	–	
h		↗	$\frac{1}{2}$	↘	
			Máx.		

A função h é estritamente crescente em $]-\pi, 0]$ e é estritamente decrescente em $\left[0, \frac{\pi}{2}\right[$. Tem um máximo relativo igual a $\frac{1}{2}$ para $x = 0$.

11.2. Uma equação da reta tangente ao gráfico de h no ponto de abcissa $x = \frac{-\pi}{2}$ é:

$$y - h\left(-\frac{\pi}{2}\right) = h'\left(-\frac{\pi}{2}\right)\left(x - \left(-\frac{\pi}{2}\right)\right)$$

$$h\left(-\frac{\pi}{2}\right) = \frac{\cos\left(-\frac{\pi}{2}\right)}{1+\cos\left(-\frac{\pi}{2}\right)} = \frac{0}{1+0} = 0$$

$$h'\left(-\frac{\pi}{2}\right) = \frac{-\sin\left(-\frac{\pi}{2}\right)}{\left(1+\cos\left(-\frac{\pi}{2}\right)\right)^2} = \frac{-(-1)}{(1+0)^2} = 1$$

Assim: $y - 0 = 1\left(x + \frac{\pi}{2}\right) \Leftrightarrow y = x + \frac{\pi}{2}$

A equação reduzida pedida é, portanto, $y = x + \frac{\pi}{2}$.

12.1. Para $x \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$, tem-se:

$$f'(x) = (\tan x)' = \frac{1}{\cos^2 x}$$

$$f''(x) = \left(\frac{1}{\cos^2 x}\right)' = \frac{(1)'\cos^2 x - 1(\cos^2 x)'}{(\cos^2 x)^2} =$$

$$= \frac{0 - 2\cos x(-\sin x)}{\cos^4 x} = \frac{2\sin x \cos x}{\cos^4 x} = \frac{\sin(2x)}{\cos^4 x}$$

Portanto, $\forall x \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$, $f''(x) = \frac{\sin(2x)}{\cos^4(x)}$

12.2. $f''(x) = 0 \Leftrightarrow \frac{\sin(2x)}{\cos^4(x)} = 0 \Leftrightarrow \sin(2x) = 0 \wedge x \in D_f \Leftrightarrow$

$$\Leftrightarrow 2x = k\pi, k \in \mathbb{Z} \wedge x \in D_f \Leftrightarrow$$

$$\Leftrightarrow x = \frac{k\pi}{2}, k \in \mathbb{Z} \wedge x \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[\Leftrightarrow x = \pi$$

x	$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$
f''		+	0	–	
f		∪		∩	
			P.I.		

O gráfico de f tem a concavidade voltada para cima em $\left]\frac{\pi}{2}, \pi\right[$ e voltada para baixo em $\left]\pi, \frac{3\pi}{2}\right[$.

Tem um único ponto de inflexão cuja abcissa é $x = \pi$.

13.1. A função g é contínua no intervalo $]0, 2\pi[$ pois é definida pela diferença e quociente defunções contínuas (função cosseno e função polinomial).

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{1}{1-\cos(x)} = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to 2\pi^-} g(x) = \lim_{x\to 2\pi^-} \frac{1}{1-\cos(x)} = \frac{1}{0^+} = +\infty$$

Portanto, as retas de equação $x = 0$ e $x = 2\pi$ são assíntotas ao gráfico de g .

O gráfico de g não tem assíntotas não verticais, já que o domínio é um conjunto limitado.

13.2. Para $x \in]0, 2\pi[$, tem-se:

$$g'(x) = \left(\frac{1}{1-\cos(x)}\right)' = \frac{(1)'(1-\cos x) - 1(1-\cos x)'}{(1-\cos x)^2} =$$

$$= \frac{0 - (-(-\sin x))}{(1-\cos x)^2} = \frac{-\sin x}{(1-\cos x)^2}$$

$$g'(x) = 0 \Leftrightarrow \frac{-\sin(x)}{(1-\cos(x))^2} = 0 \Leftrightarrow -\sin(x) = 0 \wedge x \in D_f \Leftrightarrow$$

$$\Leftrightarrow \sin(x) = 0 \wedge x \in]0, 2\pi[\Leftrightarrow x = \pi$$

x	0		π		2π
g'		–	0	+	
g		↘		↗	
			Mín.		

A função g tem, na realidade, um único mínimo, igual a $\frac{1}{2}$ para $x = \pi$.

14.1. $\lim_{x\to 0} \frac{f(x)-1}{x} = \lim_{x\to 0} \frac{1 - x + \sin(2x) - 1}{x} = \lim_{x\to 0} \frac{-x + \sin(2x)}{x} =$

$$= \lim_{x\to 0} \frac{-x}{x} + \lim_{x\to 0} \frac{\sin(2x)}{x} = -1 + 2\lim_{x\to 0} \frac{\sin(2x)}{2x} =$$

$$= -1 + 2\lim_{y\to 0} \frac{\sin y}{y} = -1 + 2 \times 1 = 1$$

$y = 2x$
Se $x \to 1, y \to 0$

14.2. Para $x \in [0, \pi[$, tem-se:

$$f'(x) = (1 - x + \sin(2x))' = -1 + 2\cos(2x)$$

$$f'(x) = 0 \Leftrightarrow -1 + 2\cos(2x) = 0 \wedge x \in [0, \pi[\Leftrightarrow$$

$$\Leftrightarrow \cos(2x) = \frac{1}{2} \wedge 2x \in [0, 2\pi[\Leftrightarrow$$

$$\Leftrightarrow 2x = \frac{\pi}{3} \vee 2x = 2\pi - \frac{\pi}{3} \Leftrightarrow x = \frac{\pi}{6} \vee x = \frac{5\pi}{6}$$

x	0		$\frac{\pi}{6}$		$\frac{5\pi}{6}$		π
f'	+	+	0	–	0	+	
f		↗		↘		↗	
	Mín.		Máx.		Mín.		

$$f\left(\frac{\pi}{6}\right) = 1 - \frac{\pi}{6} + \sin\left(2 \times \frac{\pi}{6}\right) = 1 - \frac{\pi}{6} + \sin\left(\frac{\pi}{3}\right) = 1 - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

$$f(0) = 1 - 0 + \sin(2 \times 0) = 1 + \sin(0) = 1 + 0 = 1$$

$$f\left(\frac{5\pi}{6}\right) = 1 - \frac{5\pi}{6} + \sin\left(2 \times \frac{5\pi}{6}\right) = 1 - \frac{5\pi}{6} + \sin\left(\frac{5\pi}{3}\right) =$$

$$= 1 - \frac{5\pi}{6} + \left(-\sin\left(\frac{\pi}{3}\right)\right) = 1 - \frac{5\pi}{6} - \frac{\sqrt{3}}{2}$$

Assim, $f\left(\frac{\pi}{6}\right)$ é o máximo absoluto e $f\left(\frac{5\pi}{6}\right)$ é o mínimo absoluto, logo:

$$M - m = 1 - \frac{\pi}{6} + \frac{\sqrt{3}}{2} - \left(1 - \frac{5\pi}{6} - \frac{\sqrt{3}}{2}\right) =$$

$$= -\frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{5\pi}{6} + \frac{\sqrt{3}}{2} = \frac{2\pi}{3} + \sqrt{3}$$

14.3. Para $x \in [0, \pi[$, tem-se:

$$f''(x) = (-1 + 2\cos(2x)) = 0 + 2(-2\sin(2x)) = -4\sin(2x)$$

$$f''(x) \Leftrightarrow -4\sin(2x) = 0 \Leftrightarrow \sin(2x) = 0 \Leftrightarrow$$

$$\Leftrightarrow 2x = k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{k\pi}{2}, k \in \mathbb{Z}$$

Como $x \in [0, \pi[$, tem-se que $f''(x) = 0 \Leftrightarrow x = 0 \vee x = \frac{\pi}{2}$

x	0		$\frac{\pi}{2}$		π
f''	0	–	0	+	
f	1	∩		∪	
			P.I.		

O gráfico de f tem um único ponto de inflexão de abcissa $x = \frac{\pi}{2}$.

15.1. A função g é contínua em $]\pi, 2\pi[$ pois é definida pela potência e soma de duas funções contínuas (seno e cosseno).

Logo, a função g contínua em $\left[\frac{7\pi}{6}, \frac{4\pi}{3}\right]$.

$$g\left(\frac{7\pi}{6}\right) = \sin^2\left(\frac{7\pi}{6}\right) + \cos\left(\frac{7\pi}{6}\right) =$$
$$= \left(\sin\left(\pi + \frac{\pi}{6}\right)\right)^2 + \cos\left(\pi + \frac{\pi}{6}\right) =$$
$$= \left(-\sin\frac{\pi}{6}\right)^2 - \cos\frac{\pi}{6} = \left(-\frac{1}{2}\right)^2 - \frac{\sqrt{3}}{2} =$$
$$= \frac{1}{4} - \frac{\sqrt{3}}{2} = \frac{1-2\sqrt{3}}{4} < 0$$
$$g\left(\frac{4\pi}{3}\right) = \sin^2\frac{4\pi}{3} + \cos\left(\frac{4\pi}{3}\right) =$$
$$= \left(\sin\left(\pi + \frac{\pi}{3}\right)\right)^2 + \cos\left(\pi + \frac{\pi}{3}\right) =$$
$$= \left(-\sin\frac{\pi}{3}\right)^2 - \cos\frac{\pi}{3} = \left(-\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{2} = \frac{1}{4} > 0$$

Como g é contínua no intervalo $\left[\frac{7\pi}{6}, \frac{4\pi}{3}\right]$ e $g\left(\frac{7\pi}{6}\right) \times g\left(\frac{4\pi}{3}\right) < 0$, podemos então concluir, pelo Teorema de Bolzano, que a função g tem, pelo menos, um zero no intervalo $\left]\frac{7\pi}{6}, \frac{4\pi}{3}\right[$.

15.2. O gráfico de g admite uma reta tangente horizontal num ponto em que a função derivada de g se anule.

$\forall x \in]\pi, 2\pi[$, tem-se:

$$g'(x) = \left(\sin^2 x + \cos x\right)' = 2\sin x(\sin x)' - \sin x =$$
$$= 2\sin x\cos x - \sin x$$
$$g'(x) = 0 \Leftrightarrow 2\sin x\cos x - \sin x = 0 \Leftrightarrow$$
$$\Leftrightarrow \sin x(2\cos x - 1) = 0 \Leftrightarrow \sin x = 0 \vee 2\cos x - 1 = 0 \Leftrightarrow$$
$$\Leftrightarrow \sin(x) = 0 \vee \cos(x) = \frac{1}{2}$$

Como $x \in]\pi, 2\pi[$: $g'(x) = 0 \Leftrightarrow x = \frac{5\pi}{3}$

A função derivada de g anula-se para $x = \frac{5\pi}{3}$, pelo que este valor é a abcissa do ponto de tangência.
Determinemos a sua ordenada:

$$g\left(\frac{5\pi}{3}\right) = \sin^2\frac{5\pi}{3} + \cos\frac{5\pi}{3} = \left(\sin\frac{5\pi}{3}\right)^2 + \cos\left(2\pi - \frac{\pi}{3}\right) =$$
$$= \left(\sin\left(2\pi - \frac{\pi}{3}\right)\right)^2 + \cos\left(-\frac{\pi}{3}\right) = \left(\sin\left(-\frac{\pi}{3}\right)\right)^2 + \cos\frac{\pi}{3}$$
$$= \left(-\sin\frac{\pi}{3}\right)^2 + \frac{1}{2} = \left(-\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

A abcissa pedida é $x = \frac{5\pi}{3}$ e a equação $y = \frac{5}{4}$.

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16.1. $x(0) = 6\cos\left(\frac{\pi}{2} \times 0 + \pi\right) - 1 = 6\cos(\pi) - 1 =$

$= 6 \times (-1) - 1 = -6 - 1 = -7$.

$$x(3) = 6\cos\left(\frac{\pi}{2} \times 3 + \pi\right) - 1 = 6\cos\left(\frac{5\pi}{2}\right) - 1 =$$
$$= 6\cos\left(\frac{\pi}{2}\right) - 1 = 6 \times 0 - 1 = -1$$

Logo, a abcissa do ponto P nos instantes $t = 0$ e $t = 3$ é, respetivamente, -7 e -1.

16.2. A amplitude do ponto P é 6.

16.3. Período: $T = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \times \frac{2}{\pi} = 4$

Frequência: $f = \frac{1}{T} = \frac{1}{4}$

17.1. A amplitude é 4.

17.2. O período é $\frac{2\pi}{\frac{\pi}{6}} = 12$ e a frequência é $\frac{1}{12}$.

17.3. A velocidade média da partícula nos dois primeiros segundos é dada por

$$\frac{x(2) - x(0)}{2 - 0} = \frac{4\cos\left(\frac{2\pi}{6} + \pi\right) - 4\cos\left(\frac{\pi \times 0}{6} + \pi\right)}{2} =$$
$$= \frac{4\cos\left(\pi + \frac{\pi}{3}\right) - 4\cos\pi}{2} = \frac{4\left(-\cos\frac{\pi}{3}\right) - 4 \times (-1)}{2} =$$
$$= \frac{4\left(-\frac{1}{2}\right) + 4}{2} = \frac{-2 + 4}{2} = 1$$

17.4. A velocidade da partícula no instante $t = 4$ é igual a $x'(4)$.

$$x'(t) = \left(4\cos\left(\frac{\pi t}{6} + \pi\right)\right)' =$$
$$= 4 \times \left[-\left(\frac{\pi t}{6} + \pi\right)' \sin\left(\frac{\pi t}{6} + \pi\right)\right] =$$
$$= 4\left(-\frac{\pi}{6}\sin\left(\frac{\pi t}{6} + \pi\right)\right)$$

Logo, $x'(4) = 4\left(-\frac{\pi}{6}\sin\left(\frac{4\pi}{6} + \pi\right)\right) = -\frac{2\pi}{3}\sin\left(\pi + \frac{2\pi}{3}\right) =$

$$= -\frac{2\pi}{3}\left(-\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{2\pi}{3}\left(-\sin\left(\pi - \frac{\pi}{3}\right)\right) =$$
$$= \frac{-2\pi}{3}\left(-\sin\frac{\pi}{3}\right) =$$
$$= -\frac{2\pi}{3}\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}\pi}{3}$$

A velocidade pedida é $\frac{\sqrt{3}\pi}{3}$ m/s.

17.5. A aceleração da partícula no instante $t = 2$ é dada por $x''(2)$.

$$x''(t) = \left(-\frac{2\pi}{3}\sin\left(\frac{\pi t}{6} + \pi\right)\right)' = -\frac{2\pi}{3}\left(\sin\left(\frac{\pi t}{6} + \pi\right)\right)' =$$

$$= -\frac{2\pi}{3}\left(\frac{\pi t}{6} + \pi\right)' \cos\left(\frac{\pi t}{6} + \pi\right) =$$

$$= -\frac{2\pi}{3}\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi t}{6} + \pi\right) =$$

$$= -\frac{\pi^2}{9}\cos\left(\frac{\pi t}{6} + \pi\right)$$

Assim,

$$x''(2) = -\frac{\pi^2}{9}\cos\left(\frac{2\pi}{6} + \pi\right) = -\frac{\pi^2}{9}\cos\left(\pi + \frac{\pi}{3}\right) =$$

$$= -\frac{\pi^2}{9}\left(-\cos\left(\frac{\pi}{3}\right)\right) = -\frac{\pi^2}{9}\left(-\frac{1}{2}\right) = \frac{\pi^2}{18}$$

A aceleração da partícula no instante $t = 2$ é, portanto, $\frac{\pi}{18}$.

18.
- Zeros

$$x(t) = 0 \Leftrightarrow 2\cos\left(\frac{\pi t}{4} + \pi\right) = 0 \Leftrightarrow \cos\left(\frac{\pi t}{4} + \pi\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\pi t}{4} + \pi = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \frac{\pi t}{4} = -\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \pi t = -2\pi + 4k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow t = -2 + 4k,\ k \in \mathbb{Z}$$

Como $t \in [0, 8[$, tem-se que $x(t) = 0 \Leftrightarrow t = 2 \vee t = 6$

- Contradomínio

$$\forall t \in [0, 8[,\ -1 \le \cos\left(\frac{\pi t}{4} + \pi\right) \le 1$$

Ou seja, $-2 \le 2\cos\left(\frac{\pi t}{4} + \pi\right) \le 2$, pelo que o contradomínio é $[-2, 2]$.

- Maximizantes

$$x(t) = 2 \Leftrightarrow 2\cos\left(\frac{\pi t}{4} + \pi\right) = 2 \Leftrightarrow \cos\left(\frac{\pi t}{4} + \pi\right) = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{\pi t}{4} + \pi = 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow \frac{\pi t}{4} = -\pi + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \pi t = -4\pi + 8k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow t = -4 + 8k,\ k \in \mathbb{Z}$$

Como $t \in [0, 8[$ tem-se que $t = 4$.

- Minimizantes

$$x(t) = -2 \Leftrightarrow 2\cos\left(\frac{\pi t}{4} + \pi\right) = -2 \Leftrightarrow \cos\left(\frac{\pi t}{4} + \pi\right) = -1 \Leftrightarrow$$

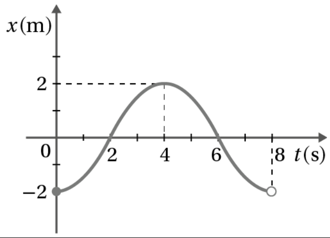
$$\Leftrightarrow \frac{\pi t}{4} + \pi = \pi + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \frac{\pi t}{4} = 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow t = 8k,\ k \in \mathbb{Z}$$

Como $t \in [0, 8[$, tem-se que $t = 0$.

- Esboço do gráfico



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19.1. Para $t \in [0, 3[$, tem-se:

$$-1 \le \cos\left(\pi t + \frac{\pi}{4}\right) \le 1, \text{ portanto,}$$

$$-3 \le 3\cos\left(\pi t + \frac{\pi}{4}\right) \le 3 \Leftrightarrow 1 \le 3\cos\left(\pi t + \frac{\pi}{4}\right) \le 7$$

Logo, a distância máxima e mínima do corpo C ao solo é, respetivamente, 7 metros e 1 metro.

19.2. A amplitude é 3.

19.3. O período é $T = \frac{2\pi}{\pi} = 2$ e a frequência é $f = \frac{1}{T} = \frac{1}{2}$

19.4. A fase é $\frac{\pi}{4}$.

19.5.
- Contradomínio

$[1, 7]$

- Maximizantes

$$D(t) = 7 \Leftrightarrow 3\cos\left(\pi t + \frac{\pi}{4}\right) + 4 = 7 \Leftrightarrow$$

$$\Leftrightarrow \cos\left(\pi t + \frac{\pi}{4}\right) = 1 \Leftrightarrow$$

$$\Leftrightarrow \pi t + \frac{\pi}{4} = 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \pi t = -\frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \pi t = -\frac{\pi}{4} + 2k\pi,\ \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow t = -\frac{1}{4} + 2k,\ k \in \mathbb{Z}$$

Como $t \in [0, 3[$, tem-se que $t = \frac{7}{4}$

- Minimizantes

$$D(t) = 1 \Leftrightarrow 3\cos\left(\pi t + \frac{\pi}{4}\right) + 4 = 1$$

$$\Leftrightarrow \cos\left(\pi t + \frac{\pi}{4}\right) + 4 = 1$$

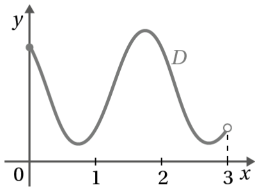
$$\Leftrightarrow \pi t + \frac{\pi}{4} = \pi + 2k\pi,\ k \in \mathbb{Z}$$

$$\Leftrightarrow \pi t = \frac{3\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$$

$$\Leftrightarrow t = \frac{3}{4} + 2k,\ k \in \mathbb{Z}$$

Como $t \in [0, 3[$, tem-se que $t = \frac{3}{4} \vee t = \frac{11}{4}$

- Esboço do gráfico



19.6. $D(t)=4 \Leftrightarrow 3\cos\left(\pi t+\frac{\pi}{4}\right)+4=4 \Leftrightarrow 3\cos\left(\pi t+\frac{\pi}{4}\right)=0 \Leftrightarrow$

$\Leftrightarrow \pi t+\frac{\pi}{4}=\frac{\pi}{2}+k\pi, k\in\mathbb{Z} \Leftrightarrow \pi t=\frac{\pi}{4}+k\pi, k\in\mathbb{Z} \Leftrightarrow$

$\Leftrightarrow t=\frac{1}{4}+k, k\in\mathbb{Z}$

Assim, o corpo C está à distância de 4 metros do solo nos instantes, em segundos, $t=\frac{1}{4}$, $t=\frac{5}{4}$ e $t=\frac{9}{4}$.

20. $f(t)=A\cos(wt+\varphi)$

20.1. $A=8$; $T=2$, pelo que $w=\frac{2\pi}{T}$, isto é, $w=\frac{2\pi}{2}=\pi$.

$f(t)=8\cos(\pi t+\varphi)$

Como $f\left(\frac{1}{2}\right)=8$, vem

$8\cos\left(\frac{\pi}{2}+\varphi\right)=8 \wedge \varphi\in[0, 2\pi[\Leftrightarrow$

$\Leftrightarrow \cos\left(\frac{\pi}{2}+\varphi\right)=1 \wedge \varphi\in[0, 2\pi[\Leftrightarrow$

$\Leftrightarrow \frac{\pi}{2}+\varphi=2k\pi, k\in\mathbb{Z} \wedge \varphi\in[0, 2\pi]$

$\Leftrightarrow \varphi=-\frac{\pi}{2}+2k\pi, k\in\mathbb{Z} \wedge \varphi\in[0, 2\pi]$

$\Leftrightarrow \varphi=\frac{3\pi}{2}$

Assim, $A=8$; $w=\pi$; T = 2 e $\varphi=\frac{3\pi}{2}$.

20.2. $f(t)=8\cos\left(\pi t+\frac{3\pi}{2}\right)$, para $t\in[0, 4]$.

20.3. $f(t)=4 \wedge t\in[0, 4] \Leftrightarrow$

$\Leftrightarrow 8\cos\left(\pi t+\frac{3\pi}{2}\right)=4 \wedge t\in[0, 4] \Leftrightarrow$

$\Leftrightarrow \cos\left(\pi t+\frac{3\pi}{2}\right)=\frac{1}{2} \wedge t\in[0, 4] \Leftrightarrow$

$\Leftrightarrow \left(\pi t+\frac{3\pi}{2}=\frac{\pi}{3}+2k\pi \vee \pi t+\frac{3\pi}{2}=-\frac{\pi}{3}+2k\pi, k\in\mathbb{Z}\right) \wedge$

$\wedge t\in[0, 4] \Leftrightarrow$

$\Leftrightarrow \left(\pi t=-\frac{7\pi}{6}+2k\pi \vee \pi t=-\frac{11\pi}{6}+2k\pi, k\in\mathbb{Z}\right) \wedge t\in[0, 4] \Leftrightarrow$

$\Leftrightarrow \left(t=-\frac{7\pi}{6}+2k \vee t=-\frac{11}{6}+2k, k\in\mathbb{Z}\right) \wedge t\in[0, 4] \Leftrightarrow$

$\Leftrightarrow t=\frac{1}{6} \vee t=\frac{5}{6} \vee t=\frac{13}{6} \vee t=\frac{17}{6}$

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21.1. Tem-se $x(t)=-3\sin(\pi t) \Leftrightarrow x(t)=3\cos\left(\pi t+\frac{\pi}{2}\right)$.

Portanto, trata-se de um oscilador harmónico já que, é dada por uma expressão da forma $x(t)=A\cos(wt+\varphi)$, onde

$A>0, w>0$ e $\varphi\in[0, 2\pi[$.

21.2. Amplitude = $A=3$

Período = $T=\frac{2\pi}{w}=\frac{2\pi}{\pi}=2$

Frequência = $\frac{1}{T}=\frac{1}{2}$

Ângulo de fase = $\frac{\pi}{2}$

21.3. Determinemos uma expressão da função x'' .

$x'(t)=\left(3\cos\left(\pi t+\frac{\pi}{2}\right)\right)'=$

$=3\left[-\left(\pi t+\frac{\pi}{2}\right)'\sin\left(\pi t+\frac{\pi}{2}\right)\right]=$

$=-3\pi\sin\left(\pi t+\frac{\pi}{2}\right)$

$x''(t)=\left(-3\pi\sin\left(\pi t+\frac{\pi}{2}\right)\right)'=$

$=-3\pi\left(\pi t+\frac{\pi}{2}\right)'\cos\left(\pi t+\frac{\pi}{2}\right)=$

$=-3\pi\left(\pi\cos\left(\pi t+\frac{\pi}{2}\right)\right)=-3\pi^2\cos\left(\pi t+\frac{\pi}{2}\right)=$

$=3\pi^2\sin(\pi t)$

Logo, $x''(t)=3\pi^2\cos\sin(\pi t)$.

Assim, vem que:

$x''(t)=-k\times x(t) \Leftrightarrow 3\pi^2\sin(\pi t)=-k\times(-3\sin(\pi t)) \Leftrightarrow$

$\Leftrightarrow 3\pi^2=3k \Leftrightarrow$

$\Leftrightarrow k=\pi^2$

Portanto, $k=\pi^2$.

22.1. Tem-se

$x(t)=8\sin\left(2t+\frac{\pi}{2}\right) \Leftrightarrow x(t)=8\cos(2t) \Leftrightarrow$

$\Leftrightarrow x(t)=8\cos(2t+0)$

Portanto, trata-se de um oscilador harmónico já que, é dada por uma expressão da forma $x(t)=A\cos(wt+\varphi)$, onde

$A>0, w>0$ e $\varphi\in[0, 2\pi[$.

22.2. Amplitude = $A=8$

Período = $T=\frac{2\pi}{2}=\pi$

Frequência = $f=\frac{1}{T}=\frac{1}{\pi}$

Ângulo de fase = 0

23.1. $x(t) = \cos t - \sqrt{3}\sin t =$

$= 2\left(\frac{1}{2}\cos t - \frac{\sqrt{3}}{2}\sin t\right) =$

$= 2\left(\cos\frac{\pi}{3}\cos t - \sin\frac{\pi}{3}\sin t\right) =$

$= 2\cos\left(\frac{\pi}{3} + t\right)$

23.2. $x(t) = 0 \land t \geq 0 \Leftrightarrow 2\cos\left(\frac{\pi}{3} + t\right) = 0 \land t \geq 0 \Leftrightarrow$

$\Leftrightarrow \cos\left(\frac{\pi}{3} + t\right) = 0 \land t \geq 0 \Leftrightarrow$

$\Leftrightarrow \frac{\pi}{3} + t = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \land t \geq 0 \Leftrightarrow$

$\Leftrightarrow t = \frac{\pi}{2} - \frac{\pi}{3} + k\pi, k \in \mathbb{Z} \land t \geq 0 \Leftrightarrow$

$\Leftrightarrow t = \frac{\pi}{6} + k\pi, k \in \mathbb{Z} \land t \geq 0 \Leftrightarrow t = \frac{\pi}{6} + k\pi, k \in \mathbb{Z}_0^+$

23.3. Determinemos uma expressão da função x''.

$x'(t) = \left(2\cos\left(\frac{\pi}{3} + t\right)\right)' =$

$= 2\left(-\left(\frac{\pi}{3} + t\right)'\sin\left(\frac{\pi}{3} + t\right)\right) =$

$= -2\sin\left(\frac{\pi}{3} + t\right)$

$x''(t) = \left(-2\sin\left(\frac{\pi}{3} + t\right)\right)' =$

$= -2\left(\left(\frac{\pi}{3} + t\right)'\cos\left(\frac{\pi}{3} + t\right)\right) =$

$= -2\cos\left(\frac{\pi}{3} + t\right)$

Assim, vem que:

$x''(t) = -k \times x(t) \Leftrightarrow -2\cos\left(\frac{\pi}{3} + t\right) = -k \times 2\cos\left(\frac{\pi}{3} + t\right)$

$\Leftrightarrow -1 = -k \Leftrightarrow k = 1$

Portanto, $k = 1$.

24.1. $f(x) = a + b\sin(cx + d), b \neq 0$ e $c > 0$

Para todo $x \in \mathbb{R}$

$f(x + P_0) = f(x) \Leftrightarrow$

$\Leftrightarrow a + b\sin\left[c(x + P_0) + d\right] = \sin(cx + d)$ $\quad b \neq 0$

$\Leftrightarrow \sin(cx + d + cP_0) = \sin(cx + d)$

Como a função seno é periódica de período fundamental 2π vem

$cP_0 = 2\pi \Leftrightarrow P_0 = \frac{2\pi}{c}(c > 0)$

Logo, f é periódica de período fundamental $\frac{2\pi}{c}$.

24.2. A função definida por $\sin(cx + d)$, com $x \in \mathbb{R}$, tem contradomínio $[-1, 1]$.

Portanto, se $b > 0$, a função definida por $b\sin(cx + d)$ tem contradomínio $[-b, b]$, donde vem que o contradomínio da função definida por $a + b\sin(cx + d)$ é $[a - b, a + b]$.

25.1. O período positivo mínimo da função g é $\frac{2\pi}{4} = \frac{\pi}{2}$.

25.2. O contradomínio da função g é $[2 - 3, 2 + 3]$, isto é, $[-1, 5]$.

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26. $f(x) = a + b\sin(cx + d)$

O contradomínio da função h é um intervalo de números reais do tipo $[a - b, a + b]$. Como o contradomínio é $[-3, 2]$, tem-se que:

$\begin{cases} a - b = -3 \\ a + b = 2 \end{cases} \Leftrightarrow \begin{cases} a = -3 + b \\ -3 + b + b = 2 \end{cases} \Leftrightarrow \begin{cases} a = -3 + b \\ 2b = 5 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} a = -3 + \frac{5}{2} \\ b = \frac{5}{2} \end{cases} \Leftrightarrow \begin{cases} a = -\frac{1}{2} \\ b = \frac{5}{2} \end{cases}$

Por outro lado, o período mínimo positivo da função h é 6, portanto, $\frac{2\pi}{c} = 6 \Leftrightarrow c = \frac{2\pi}{6} \Leftrightarrow c = \frac{\pi}{3}$

Assim, h é definida por uma expressão da forma:

$h(x) = -\frac{1}{2} + \frac{5}{2}\sin\left(\frac{\pi}{3}x + d\right)$

Como o ponto de coordenadas $\left(0, -\frac{7}{4}\right)$ pertence ao gráfico de h, vem que:

$h(0) = -\frac{7}{4} \Leftrightarrow -\frac{1}{2} + \frac{5}{2}\sin\left(\frac{\pi}{3} \times 0 + d\right) = -\frac{7}{4} \Leftrightarrow$

$\Leftrightarrow \frac{5}{2}\sin d = -\frac{7}{4} + \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \frac{5}{2}\sin d = -\frac{5}{4} \Leftrightarrow \sin d = -\frac{5}{4} \times \frac{2}{5} \Leftrightarrow \sin d = -\frac{1}{2} \Leftrightarrow$

$\Leftrightarrow d = -\frac{\pi}{6} + 2k\pi \lor d = \frac{7\pi}{6} + 2k\pi, k \in \mathbb{Z}$

Portanto, um valor possível para d é $-\frac{\pi}{6}$.

Assim, a função h pode ser definida por

$h(x) = -\frac{1}{2} + \frac{5}{2}\sin\left(\frac{\pi}{3}x - \frac{\pi}{6}\right)$

27.1. $f(x) = 3\cos\left(x + \frac{\pi}{4}\right) + 5$

- $D_f = \mathbb{R}$
- $D'_f = [-3 + 5, 3 + 5] = [2, 8]$
- Período: $\frac{2\pi}{1} = 2\pi$

- Zeros: não tem, pois $D'_f = [2, 8]$
- Maximizantes

$f(x) = 8 \Leftrightarrow 3\cos\left(x + \frac{\pi}{4}\right) + 5 = 8 \Leftrightarrow$

$\Leftrightarrow \cos\left(x + \frac{\pi}{4}\right) = 1 \Leftrightarrow$

$\Leftrightarrow x + \frac{\pi}{4} = 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x = -\frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$

- Minimizantes

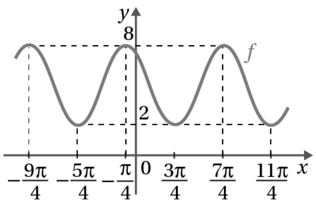
$f(x) = 2 \Leftrightarrow 3\cos\left(x + \frac{\pi}{4}\right) + 5 = 2 \Leftrightarrow$

$\Leftrightarrow \cos\left(x + \frac{\pi}{4}\right) = -1 \Leftrightarrow$

$\Leftrightarrow x + \frac{\pi}{4} = \pi + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x = \frac{3\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$

- Esboço do gráfico



27.2. ▪ $D_g = \left\{x \in \mathbb{R} : \frac{x\pi}{3} + \frac{\pi}{9} \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\} =$

$= \left\{x \in \mathbb{R} : \frac{x\pi}{3} \neq \frac{\pi}{2} - \frac{\pi}{9} + k\pi,\ k \in \mathbb{Z}\right\} =$

$= \left\{x \in \mathbb{R} : \frac{x\pi}{3} \neq \frac{7\pi}{18} + k\pi,\ k \in \mathbb{Z}\right\} =$

$= \left\{x \in \mathbb{R} : x \neq \frac{7}{6} + 3k,\ k \in \mathbb{Z}\right\} =$

$= \mathbb{R} \setminus \left\{\frac{7}{6} + 3k,\ k \in \mathbb{Z}\right\}$

- $D'_g = \mathbb{R}$
- Zeros

$g(x) = 0 \Leftrightarrow -10 + 10\tan\left(\frac{x\pi}{3} + \frac{\pi}{9}\right) = 0 \Leftrightarrow$

$\Leftrightarrow \tan\left(\frac{x\pi}{3} + \frac{\pi}{9}\right) = 1 \Leftrightarrow$

$\Leftrightarrow \frac{x\pi}{3} + \frac{\pi}{9} = \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

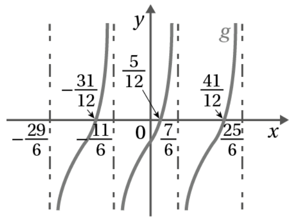
$\Leftrightarrow \frac{x\pi}{3} = \frac{\pi}{4} - \frac{\pi}{9} + k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow \frac{x\pi}{3} = \frac{5\pi}{36} + k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow x = \frac{5}{12} + 3k,\ k \in \mathbb{Z}$

- Maximizantes: Não tem
- Minimizantes: Não tem

- Esboço do gráfico



28. Seja f a função pedida: $f(t) = a + b\cos(kt + c)$

- O período de f é $P_0 = 16(8+8)$

$\frac{2\pi}{8} = 16 \Leftrightarrow k = \frac{\pi}{8}$ (opções B e C)

- Sabemos que $f(8) = 12$

Em B, $f(8) = 10 - 2\cos\left(\frac{\pi}{8} \times 8\right) = 10 - 2 \times \cos\pi =$

$= 10 + 2 = 12$

Em C, $f(8) = 10 + 2\cos\left(\frac{\pi}{8} \times 8\right) = 10 - 2 = 8$

Resposta: **(B)**

Ficha de teste 6

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1. O triângulo $[OAC]$ é isósceles $\left(\overline{OA} = \overline{OC}\right)$

Assim, como P é o ponto médio de $[AC]$,

$A\hat{O}P = \frac{A\hat{O}C}{2} = \frac{x}{2}$

Como o triângulo $[APO]$ é retângulo em P, e $\overline{OA} = 1$ vem

$\overline{AP} = \sin\frac{x}{2}$ e $\overline{OP} = \cos\frac{x}{2}$

$A_{[ACO]} = \frac{2\overline{AP} \times \overline{OP}}{2} = \overline{AO} \times \overline{OP} = \sin\frac{x}{2}\cos\frac{x}{2}$

Resposta: **(C)**

2. $\cos\left(x + \frac{5\pi}{6}\right)\cos\left(x - \frac{2\pi}{3}\right) + \sin\left(x + \frac{5\pi}{6}\right)\sin\left(x - \frac{2\pi}{3}\right) =$

$= \cos\left[\left(x + \frac{5\pi}{6}\right) - \left(x - \frac{2\pi}{3}\right)\right] = \cos\left(x + \frac{5\pi}{6} - x + \frac{2\pi}{3}\right) =$

$= \cos\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right) = \cos\left(\frac{9\pi}{6}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$

Resposta: **(B)**

3. $\lim\limits_{x \to 0} \frac{\sin(\alpha + x) - \sin(\alpha - x)}{x}$

$= \lim\limits_{x \to 0} \frac{\sin\alpha\cos x + \sin x\cos\alpha - (\sin\alpha\cos x - \sin x\cos\alpha)}{x} =$

$= \lim\limits_{x \to 0} \frac{2\sin x\cos\alpha}{x} = \lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right) \times \lim\limits_{x \to 0}(2\cos\alpha) =$

$1 + \tan^2 = \frac{1}{\cos^2\alpha}$; $\tan\alpha = \frac{1}{3}$

$1 + \left(\frac{1}{3}\right)^2 = \frac{1}{\cos^2\alpha} \Leftrightarrow 1 + \frac{1}{9} = \frac{1}{\cos^2\alpha} \Leftrightarrow \frac{10}{9} = \frac{1}{\cos^2\alpha} \Leftrightarrow$

$\Leftrightarrow \cos^2\alpha = \frac{9}{10} \Leftrightarrow \cos\alpha = -\sqrt{\frac{9}{10}} \vee \cos\alpha = \sqrt{\frac{9}{10}} \Leftrightarrow$

$\Leftrightarrow \cos\alpha = -\frac{3\sqrt{10}}{10} \vee \cos\alpha = \frac{3\sqrt{10}}{10}$

Como $\alpha \in \left]\pi, \frac{3\pi}{2}\right[$, $\cos\alpha < 0$, pelo que $\cos\alpha = -\frac{3\sqrt{10}}{10}$

Assim,

$$\lim_{x\to 0}\frac{\sin(\alpha + x) - \sin(\alpha - x)}{x} = 2\cos\alpha =$$

$$= 2\left(-\frac{3\sqrt{10}}{10}\right) = -\frac{6\sqrt{10}}{10} = -\frac{3\sqrt{10}}{5}$$

Resposta: **(A)**

4. Se a função g é contínua no intervalo $\left[\frac{\pi}{4}, \frac{\pi}{2}\right[$ então é contínua no ponto $\frac{\pi}{2}$ pelo que existe $\lim_{x\to\frac{\pi}{2}} g(x)$.

$$\lim_{x\to\frac{\pi}{2}^-} g(x) = \lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{\frac{\pi}{2} - x} =$$

$y = x - \frac{\pi}{2} \Leftrightarrow x = \frac{\pi}{2} + y$

Se $x \to \frac{\pi}{2}^+$, $y \to 0^+$

$$= \lim_{y\to 0^+}\frac{\cos\left(y + \frac{\pi}{2}\right)}{-y} =$$

$$= \lim_{y\to 0^+}\frac{-\sin y}{-y} = \lim_{y\to 0^+}\frac{\sin y}{y} = 1$$

$$\lim_{x\to\frac{\pi}{2}^-} g(x) = g\left(\frac{\pi}{2}\right) \Leftrightarrow 1 = k - 1 \Leftrightarrow k = 2$$

Resposta: **(D)**

5. $$f'(x) = \left(\cos^2 x^2\right)' = \left[\left(\cos x^2\right)^2\right]' = 2\cos x^2\left(\cos x^2\right)' =$$

$$= 2\cos x^2\left(-x^{2\prime}\sin x^2\right) =$$

$$= 2\cos x^2\left(-2x\sin x^2\right) = -2x \times 2\cos x^2 \sin x^2 =$$

$$= -2x\sin\left(2x^2\right)$$

Resposta: **(A)**

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6. O declive da reta r é igual a $g'\left(\frac{\pi}{6}\right)$.

$$g'(x) = \left(\sin(2x)\right)' = (2x)'\cos(2x) = 2\cos(2x).$$

Assim, tem-se que:

$$g'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{2\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) = 2\times\frac{1}{2} = 1$$

Retas paralelas têm o mesmo declive, pelo que o declive da reta s é igual a 1.

Por outro lado, esta reta passa pela origem , pelo que é definida por uma equação do tipo $y = mx$, onde m é o declive. Portanto, $y = x$ é a equação reduzida da reta s.

Resposta: **(B)**

7. Como a função é contínua em $\mathbb{R}$ então é contínua em $x = 0$.

Assim existe $\lim_{x\to 0} f(x)$,

$$\lim_{x\to 0^-}\frac{\sin(2x)}{x} = 2\lim_{x\to 0^-}\frac{\sin(2x)}{2x} =$$

$y = 2x$

Se $x \to 0$, $y \to 0$

$$= 2\lim_{y\to 0^-}\frac{\sin y}{y} = 2\times 1 = 2$$

$$\lim_{x\to 0^+}\left(\sqrt{k} + x\right) = f(0) = \sqrt{k} + 0 = \sqrt{k}.$$

Portanto, $\sqrt{k} = 2$ e como $k \in \mathbb{R}^+$, vem $k = 4$.

Resposta: **(D)**

8. ▪ Contradomínio de h

Para $x \in \mathbb{R}$:

$$-1 \le \cos\left[3\left(x + \frac{\pi}{18}\right)\right] \le 1 \Leftrightarrow$$

$$\Leftrightarrow -5 \le 5\cos\left[3\left(x + \frac{\pi}{18}\right)\right] \le 5 \Leftrightarrow$$

$$\Leftrightarrow -7 \le 5\cos\left[3\left(x + \frac{\pi}{18}\right)\right] - 2 \le 3$$

Portanto, $D'_h = [-7, 3]$, pelo que a opção (A) é rejeitada.

▪ O período da função g é $\frac{2\pi}{3}$ e não $\frac{\pi}{8}$ pelo que a opção (B) é rejeitada

• $$h\left(\frac{17\pi}{54}\right) = 5\cos\left[3\left(\frac{17\pi}{54} + \frac{\pi}{18}\right)\right] - 2 =$$

$$= 5\cos\left(\frac{17\pi}{18} + \frac{\pi}{6}\right) - 2 =$$

$$= 5\cos\left(\frac{17\pi + 3\pi}{18}\right) - 2 =$$

$$= 5\cos\frac{20\pi}{18} - 2 =$$

$$= 5\cos\frac{10\pi}{9} - 2 =$$

$$= 5\cos\left(\pi + \frac{\pi}{9}\right) - 2 =$$

$$= -5\cos\left(\frac{\pi}{9} - 2\right)$$

$$h\left(\frac{17}{54}\right) \ne 3.$$

Logo, $\frac{17\pi}{54}$ não é maximizante de h.

• $$h\left(\frac{53\pi}{18}\right) = 5\cos\left[3\left(\frac{53\pi}{18} + \frac{\pi}{18}\right)\right] - 2 =$$

$$= 5\cos\left(3\times\frac{54\pi}{18}\right) - 2 =$$

$$= 5\cos(3\times 3\pi) - 2 =$$

$$= 5\cos(8\pi + \pi) - 2 =$$

$$= 5\cos\pi - 2 =$$

$$= 5\times(-1) - 2 = -7$$

Logo, $\frac{53\pi}{18}$ é um minimizante de h.

Resposta: **(D)**

9. Por definição:

$$\lim_{h\to 0}\frac{\sin(\pi + h) - \sin(\pi)}{h} = g'(\pi), \text{ sendo } g(x) = \sin x.$$

$g'(x) = (\sin x)' = \cos x$, pelo que $g'(\pi) = \cos\pi = -1$.

Resposta: **(C)**

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10.1. $\sin(2x) + \sqrt{3}\cos(2x) = -1 \Leftrightarrow$

$$\Leftrightarrow \frac{1}{2}\sin(2x) + \frac{\sqrt{3}}{2}\cos(2x) = -\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \sin\left(\frac{\pi}{6}\right)\sin(2x) + \cos\left(\frac{\pi}{6}\right)\cos(2x) = -\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \cos(2x)\cos\left(\frac{\pi}{6}\right)+\sin(2x)\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \cos\left(2x-\frac{\pi}{6}\right)=-\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \cos\left(2x-\frac{\pi}{6}\right)=\cos\left(\frac{2\pi}{3}\right) \Leftrightarrow$$

$$\Leftrightarrow 2x-\frac{\pi}{6}=\frac{2\pi}{3}+2k\pi \vee 2x-\frac{\pi}{6}=-\frac{2\pi}{3}+2k\pi,\ k\in\mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow 2x=\frac{\pi}{6}+\frac{2\pi}{3}+2k\pi \vee 2x=\frac{\pi}{6}-\frac{2\pi}{3}+2k\pi,\ k\in\mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow 2x=\frac{5\pi}{6}+2k\pi \vee 2x=-\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x=\frac{5\pi}{12}+k\pi \vee x=-\frac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$$

10.2. $\cos(2x)+7\sin x+3=0 \Leftrightarrow$

$$\Leftrightarrow \cos^2 x-\sin^2 x+7\sin x+3=0 \Leftrightarrow$$

$$\Leftrightarrow 1-\sin^2 x-\sin^2 x+7\sin x+3=0 \Leftrightarrow$$

$$\Leftrightarrow -2\sin^2 x+7\sin x+4=0 \Leftrightarrow$$

$$\Leftrightarrow \sin x=\frac{-7\pm\sqrt{49-4(-2)4}}{2(-2)} \Leftrightarrow$$

$$\Leftrightarrow \sin(x)=\frac{-7\pm\sqrt{81}}{-4} \Leftrightarrow \sin(x)=\frac{-7\pm 9}{-4} \Leftrightarrow$$

$$\Leftrightarrow \sin(x)=4 \vee \sin x=-\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow x\in\varnothing \vee \sin(x)=-\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \sin x=-\frac{1}{2} \Leftrightarrow \sin x=\sin\left(-\frac{\pi}{6}\right) \Leftrightarrow$$

$$\Leftrightarrow x=-\frac{\pi}{6}+2k\pi \vee x=\frac{7\pi}{6}+2k\pi,\ k\in\mathbb{Z}$$

11. Tem-se que:

$$\overrightarrow{BD}\cdot\overrightarrow{DC}=\left\|\overrightarrow{BD}\right\|\times\left\|\overrightarrow{DC}\right\|\times\cos\left(\overrightarrow{BD}\hat{,}\overrightarrow{DC}\right)$$

$$\left\|\overrightarrow{DC}\right\|=\left\|\overrightarrow{AB}\right\|=\overline{AB}=2$$

$$\overline{BD}^2=\overline{AB}^2+\overline{AD}^2 \Leftrightarrow \overline{BD}^2=2^2+6^2$$

$$\Leftrightarrow \overline{BD}^2=4+36 \Leftrightarrow \overline{BD}^2=40$$

Como $\overline{BD}>0$, $\overline{BD}=\sqrt{40}=2\sqrt{10}$.

Logo, $\left\|\overrightarrow{BD}\right\|=\overline{BD}=2\sqrt{10}$.

A amplitude do ângulo formado pelos vetores $\overrightarrow{BD}$ e $\overrightarrow{DC}$ é dada por $\frac{\pi}{2}+\alpha$.

Assim, vem:

$$\overrightarrow{BD}\cdot\overrightarrow{DC}=2\sqrt{10}\times 2\times\cos\left(\frac{\pi}{2}+\alpha\right)=$$

$$=4\sqrt{10}\times(-\sin\alpha)=$$

$$=-4\sqrt{10}\sin\alpha=$$

$$=-4\sqrt{10}\left(2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right)=$$

$$=-8\sqrt{10}\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)$$

12.1. $\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x+\cos x)=0+\cos(0)=0+1=1=g(0)$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}\frac{\sin(x)-x}{\sin(2x)}\overset{\left(\frac{0}{0}\right)}{=}$$

$$=\lim_{x\to 0}\frac{\frac{\sin x}{x}-1}{\frac{\sin(2x)}{x}}=\frac{\lim_{x\to 0}\frac{\sin x}{x}-1}{2\times\lim_{x\to 0}\frac{\sin(2x)}{2x}}=$$

$$=\frac{1-1}{2\times\lim_{y\to 0}\frac{\sin y}{y}}=$$

$y=2x$ \
Se $x\to 0$, $y\to 0$

$$=\frac{0}{2\times 1}=0$$

Como $\lim_{x\to 0^+} g(x)\neq\lim_{x\to 0^-} g(x)$, não existe $\lim_{x\to 0} g(x)$ pelo que a função g não é contínua no ponto de abcissa $x=0$.

12.2. A função g é contínua no intervalo $]0,+\infty[$ pois é definida pela soma de duas funções contínuas (uma função polinomial e a função cosseno).

Como $[3,4]\subset\,]0,+\infty[$ a função g é contínua no intervalo $[3,4]$.

$g(3)=3+\cos(3)<3<\pi$ e $g(4)=4+\cos(4)\approx 3{,}34>\pi$.

Como g é contínua no intervalo $[3,4]$ e $g(3)<\pi<g(4)$, podemos então concluir, pelo Teorema de Bolzano-Cauchy, que a equação $g(x)=\pi$ tem, pelo menos, uma solução, no intervalo $]3,4[$.

13. A função h é contínua.

Assim, as retas de equação $x=-2\pi$ e $x=0$ são as únicas possíveis assíntotas verticais do gráfico de h .

$$\lim_{x\to -2\pi^+} h(x)=\lim_{x\to -2\pi^+}\frac{x-\sin(x)}{2\sin\left(\frac{x}{2}\right)}=$$

$$=\frac{-2\pi-\sin(-2\pi^+)}{2\sin\left(\frac{-2\pi^+}{2}\right)}=\frac{-2\pi-0}{0^-}=+\infty$$

$$\lim_{x\to 0^-} h(x)=\lim_{x\to 0^-}\frac{x-\sin(x)}{2\sin\left(\frac{x}{2}\right)}\overset{\left(\frac{0}{0}\right)}{=}$$

$$=\lim_{x\to 0^-}\frac{1-\frac{\sin x}{x}}{2\times\frac{\sin x}{x}}=\frac{1-\lim_{x\to 0^-}\frac{\sin x}{x}}{2\times\frac{1}{2}\times\lim_{x\to 0^-}\frac{\sin\frac{x}{2}}{\frac{x}{2}}}=$$

$$=\frac{1-1}{1\times\lim_{y\to 0^-}\frac{\sin y}{y}}=$$

$y=\frac{x}{2}$ \
Se $x\to 0^-$, $y\to 0^-$

$$=\frac{0}{1\times 1}=0$$

Como $\lim_{x\to -2\pi^+} h(x)=+\infty$ e $\lim_{x\to 0^-} h(x)=0$, apenas a reta de equação $x=-2\pi$ é assíntota ao gráfico de h .

14.1. $0{,}8\sin\left(wt+\frac{\pi}{6}\right)=0{,}8\left[\sin(wt)\cos\left(\frac{\pi}{6}\right)\cos(wt)\right]$

$=0{,}8\left[\sin(wt)\cos\frac{\pi}{6}+\sin\frac{\pi}{6}\cos(wt)\right]=$

$=\frac{4}{5}\left[\sin(wt)\times\frac{\sqrt{3}}{2}+\frac{1}{2}\cos(wt)\right]$

$=\frac{2\sqrt{3}}{5}\sin(wt)+\frac{2}{5}\cos(wt)$

14.2. $f(t)=\frac{2}{5}\cos(wt)-\frac{3}{5}\Leftrightarrow$

$\Leftrightarrow\frac{2\sqrt{3}}{5}\sin(wt)+\frac{2}{5}\cos(wt)=\frac{2}{5}\cos(wt)-\frac{3}{5}\Leftrightarrow$

$\Leftrightarrow\frac{2\sqrt{3}}{5}\sin(wt)=-\frac{3}{5}\Leftrightarrow 2\sqrt{3}\sin(wt)=-3\Leftrightarrow$

$\Leftrightarrow\sin(wt)=-\frac{3}{2\sqrt{3}}\Leftrightarrow\sin(wt)=-\frac{\sqrt{3}}{2}\Leftrightarrow$

$\Leftrightarrow\sin(wt)=\sin\left(-\frac{\pi}{3}\right)\Leftrightarrow$

$\Leftrightarrow wt=-\frac{\pi}{3}+2k\pi\vee wt=\frac{4\pi}{3}+2k\pi,\ k\in\mathbb{Z}$

$\Leftrightarrow t=-\frac{\pi}{3w}+\frac{2\pi}{w}k\vee t=\frac{4\pi}{3w}+\frac{2\pi}{w}k,\ k\in\mathbb{Z}$

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15.1. Tem-se:

$f'(x)=\lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to0}\frac{\sin^2(x+h)-\sin^2 x}{h}=$

$=\lim\limits_{h\to0}\frac{[\sin(x+h)-\sin x][\sin(x+h)+\sin x]}{h}=$

$=\lim\limits_{h\to0}[\sin(x+h)+\sin x]\times\lim\limits_{h\to0}\frac{\sin(x+h)-\sin x}{h}=$

$=2\sin x\times\lim\limits_{h\to0}\frac{\sin x\cosh+\sinh\cos x-\sin x}{h}=$

$=2\sin x\times\cos x=\sin(2x)$, dado que

$\lim\limits_{h\to0}\frac{\sin x\cosh+\sinh\cos x-\sin x}{h}=$

$=\lim\limits_{h\to0}\frac{\sin x\cos x-\sin x}{h}+\lim\limits_{h\to0}\frac{\sinh\cos x}{h}=$

$=\lim\limits_{h\to0}\frac{-\sin x(1-\cos h)}{h}+\cos x\times\lim\limits_{h\to0}\frac{\sin h}{h}=$

$=-\sin x\times\lim\limits_{h\to0}\frac{(1-\cos h)(1+\cos h)}{h}+\cos x\times1=$

$=-\sin x\times\lim\limits_{h\to0}\frac{1-\cos^2 h}{h}+\cos x=$

$=-\sin x\times\lim\limits_{h\to0}\frac{\sin^2 h}{h}+\cos x=$

$=-\sin x\times\lim\limits_{h\to0}\frac{\sin h}{h}\times\lim\limits_{h\to0}\sin h+\cos x=$

$=-\sin x\times1\times0+\cos x=$

$=\cos x$

Portanto, $f'(x)=\sin(2x)$

15.2. $f'(x)=0\Leftrightarrow\sin(2x)=0\Leftrightarrow 2x=k\pi,\ k\in\mathbb{Z}\Leftrightarrow$

$\Leftrightarrow x=\frac{k\pi}{2},\ k\in\mathbb{Z}\Leftrightarrow$

Como $x\in\left]0,\frac{3\pi}{2}\right]$, vem $x=\frac{\pi}{2}\vee x=\pi\vee x=\frac{3\pi}{2}$.

x	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$
f'		+	0	–	0	+	0
f		↗	1	↘	0	↗	1
			Máx.		Mín.		Máx.

A função f, neste intervalo, é estritamente crescente em $\left]0,\frac{\pi}{2}\right]$ e em $\left[\pi,\frac{3\pi}{2}\right]$ e é estritamente decrescente em $\left[\frac{\pi}{2},\pi\right]$. Tem dois máximos relativos, são eles $f\left(\frac{\pi}{2}\right)=1$ e $f\left(\frac{3\pi}{2}\right)=1$ e tem um mínimo relativo que é $f(\pi)=0$.

16. A reta t tem declive igual a –1, pelo que a abcissa do ponto A é a solução da equação $g'(x)=-1$, com $x\in]-\pi,\pi[$.

$g'(x)=(\sin x-\cos x)'=\cos x-(-\sin x)$

$=\cos x+\sin x$

$g'(x)=-1\Leftrightarrow\cos(x)+\sin(x)=-1\Leftrightarrow$

$\Leftrightarrow\frac{\sqrt{2}}{2}\cos x+\frac{\sqrt{2}}{2}\times\sin x=-\frac{\sqrt{2}}{2}\Leftrightarrow$

$\Leftrightarrow\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x=-\cos\left(\frac{\pi}{4}\right)\Leftrightarrow$

$\Leftrightarrow\cos\left(x-\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)$

$\Leftrightarrow\cos\left(x-\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)$

$\Leftrightarrow x-\frac{\pi}{4}=\pi-\frac{\pi}{4}+2k\pi\vee x-\frac{\pi}{4}=-\pi+\frac{\pi}{4}+2k\pi,k\in\mathbb{Z}\Leftrightarrow$

$\Leftrightarrow x=\pi+2k\pi\vee x=-\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}\Leftrightarrow$

Como $x\in]-\pi,\pi[$, tem-se que $x=-\frac{\pi}{2}$.

As coordenadas do ponto A são $\left(-\frac{\pi}{2},g\left(-\frac{\pi}{2}\right)\right)$

$g\left(-\frac{\pi}{2}\right)=\sin\left(-\frac{\pi}{2}\right)-\cos\left(-\frac{\pi}{2}\right)=-1-0=-1$

$A\left(-\frac{\pi}{2},-1\right)$.

Uma equação da reta t é:

$y-g\left(-\frac{\pi}{2}\right)=g'\left(-\frac{\pi}{2}\right)\left(x-\left(-\frac{\pi}{2}\right)\right)\Leftrightarrow$

$\Leftrightarrow y-(-1)=-1\left(x+\frac{\pi}{2}\right)\Leftrightarrow$

$\Leftrightarrow y+1=-x-\frac{\pi}{2}\Leftrightarrow$

$\Leftrightarrow y=-x-\frac{\pi}{2}-1$

Portanto, a equação reduzida da reta t é $y = -x - \frac{\pi}{2} - 1$.

17.1. Para $x \in \left]0, \pi\right[$, tem-se:

$$h'(x) = \left(x^2 + \cos(2x)\right)' = \left(x^2\right)' + \left(\cos(2x)\right)' =$$
$$= 2x - 2\sin(2x)$$

$$h''(x) = \left(2x - 2\sin(2x)\right)' = (2x)' - \left(2\sin(2x)\right)' =$$
$$= 2 - 2\left(\sin(2x)\right)' = 2 - 2\times 2\cos(2x) = 2 - 4\cos(2x)$$

$$h''(x) = 0 \Leftrightarrow 2 - 4\cos(2x) = 0 \Leftrightarrow \cos(2x) = \frac{1}{2} \Leftrightarrow$$
$$\Leftrightarrow 2x = \frac{\pi}{3} + 2k\pi \vee 2x = -\frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$
$$\Leftrightarrow x = \frac{\pi}{6} + k\pi \vee x = -\frac{\pi}{6} + k\pi,\ k \in \mathbb{Z}$$

Como $x \in \left]0, \pi\right[$ tem-se $h''(x) = 0 \Leftrightarrow x = \frac{\pi}{6} \vee x = \frac{5\pi}{6}$

x	0		$\frac{\pi}{6}$		$\frac{5\pi}{6}$		π
h''		–	0	+	0	–	
h		$\cap$		$\cup$		$\cap$	
			P.I.		P.I.		

O gráfico de h tem a concavidade voltada para cima em $\left]\frac{\pi}{6}, \frac{5\pi}{6}\right[$ e voltada para baixo em $\left]0, \frac{\pi}{6}\right[$ e em $\left]\frac{5\pi}{6}, \pi\right[$.

Tem dois pontos de inflexão de abcissas $\frac{\pi}{6}$ e $\frac{5\pi}{6}$.

17.2. A função h é contínua em $\left]0, \pi\right[$ pois é definida pela composta e soma de funções contínuas (função cosseno e funções polinomiais).

Como $\left[\frac{7}{4}, 2\right] \subset \left]0, \pi\right[$, a função h é contínua no intervalo $\left[\frac{7}{4}, 2\right]$.

$$h\left(\frac{7}{4}\right) = \left(\frac{7}{4}\right)^2 + \cos\left(2\times\frac{7}{4}\right) \approx 2{,}126$$
$$h(2) = 2^2 + \cos(2\times 2) \approx 3{,}346$$

Como h é contínua no intervalo $\left[\frac{7}{4}, 2\right]$ e $h\left(\frac{7}{4}\right) < \pi < h(2)$, podemos então concluir, pelo Teorema de Bolzano que $\exists x \in \left]\frac{7}{4}, 2\right[: h(x) = \pi$, isto é, que a equação $h(x) = \pi$ tem, pelo menos, uma solução no intervalo $\left]\frac{7}{4}, 2\right[$.

18.1. $x(t) = \sin(\pi t) - \frac{\sqrt{3}}{3}\cos(\pi t) =$

$$= -\frac{\sqrt{3}}{3}\left(\cos(\pi t) - \frac{3}{\sqrt{3}}\sin(\pi t)\right) =$$
$$= -\frac{\sqrt{3}}{3}\left(\cos(\pi t) - \sqrt{3}\sin(\pi t)\right) = \qquad \left| \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}\right.$$
$$= -\frac{\sqrt{3}}{3}\times 2\left(\frac{1}{2}\cos(\pi t) - \frac{\sqrt{3}}{2}\sin(\pi t)\right) =$$
$$= -\frac{2\sqrt{3}}{3}\left(\cos(\pi t)\cos\frac{\pi}{3} - \sin\frac{\pi}{3}\sin(\pi t)\right) =$$
$$= -\frac{2\sqrt{3}}{3}\cos\left(\pi t + \frac{\pi}{3}\right) =$$
$$= \frac{2\sqrt{3}}{3}\cos\left(\pi t + \frac{\pi}{3} + \pi\right) =$$
$$= \frac{2\sqrt{3}}{3}\cos\left(\pi t + \frac{4\pi}{3}\right)$$

Trata-se de um oscilador harmónico, uma vez que, é dada por uma expressão da forma $x(t) = A\cos(wt + \varphi)$, onde $A > 0,\ w > 0$ e $\varphi \in \left[0, 2\pi\right[$.

18.2. Amplitude: $\frac{2\sqrt{3}}{3}$

Período: $\frac{2\pi}{\pi} = 2$

Frequência: $\frac{1}{2}$

Ângulo de fase: $\frac{4\pi}{3}$

18.3. Determinemos uma expressão da função x''.

$$x'(t) = \left(\frac{2\sqrt{3}}{3}\cos\left(\pi t + \frac{4\pi}{3}\right)\right)' =$$
$$= \frac{2\sqrt{3}}{3}\left(-\left(\pi t + \frac{4\pi}{3}\right)'\sin\left(\pi t + \frac{4\pi}{3}\right)\right) =$$
$$= \frac{2\sqrt{3}}{3}\left(-\pi\sin\left(\pi t + \frac{4\pi}{3}\right)\right) =$$
$$= -\frac{2\sqrt{3}\pi}{3}\sin\left(\pi t + \frac{4\pi}{3}\right) =$$

$$x''(t) = \left(-\frac{2\sqrt{3}}{3}\pi\sin\left(\pi t + \frac{4\pi}{3}\right)\right)' =$$
$$= -\frac{2\sqrt{3}}{3}\pi\left(\left(\pi t + \frac{4\pi}{3}\right)'\cos\left(\pi t + \frac{4\pi}{3}\right)\right) =$$
$$= \frac{-2\sqrt{3}}{3}\pi\left(\pi\cos\left(\pi t + \frac{4\pi}{3}\right)\right) =$$
$$= -\frac{2\sqrt{3}\pi^2}{3}\cos\left(\pi t + \frac{4\pi}{3}\right)$$

Logo,

$$x''(t) = -k(xt) \Leftrightarrow$$
$$\Leftrightarrow -\frac{2\sqrt{3}}{3}\pi^2\cos\left(\pi t + \frac{4\pi}{3}\right) = -k\left(\frac{2\sqrt{3}}{3}\cos\left(\pi t + \frac{4\pi}{3}\right)\right) \Leftrightarrow$$
$$\Leftrightarrow \pi^2 = k$$

Portanto $k = \pi^2$.

Interpretação de $-kx(t)$: trata-se da força exercida (pela mola) sobre o ponto P («Lei de Hooke»).