Ficha para praticar 17

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1.1. z_1 imaginário puro se e só se $\text{Re}(z_1)=0 \wedge \text{Im}(z_1)\neq 0$, ou seja, $-3k=0 \Leftrightarrow k=0$

1.2. z_2 é um imaginário puro se e só se $\text{Re}(z_2)=0 \wedge \text{Im}(z_2)\neq 0$, ou seja.
$k-3=0 \wedge k^2-9\neq 0 \Leftrightarrow k=3 \wedge (k\neq -3 \wedge k\neq 3) \Leftrightarrow k\in\varnothing$,
pelo que z_2 nunca é imaginário puro.

1.3. z_3 é um imaginário puro se e só se $\text{Re}(z)=0 \wedge \text{Im}(z_3)\neq 0$, ou seja,
$k^2-k=0 \wedge k\neq 0 \Leftrightarrow k(k-1)=0 \wedge k\neq 0 \Leftrightarrow$
$\Leftrightarrow (k=0 \vee k=1) \wedge k\neq 0 \Leftrightarrow k=1$

1.4. $z_4=2k\text{i}-3+k^2\text{i}+k \Leftrightarrow z_4=(-3+k)+(2k+k^2)\text{i}$ e z_4 é um número real se e só se $\text{Im}(z_4)=0$, ou seja,
$2k+k^2=0 \Leftrightarrow k(2+k)=0 \Leftrightarrow k=0 \vee k=-2$.

1.5. z_5 é um número real negativo se e só se
$\text{Re}(z_5)<0 \wedge \text{Im}(z_5)=0$, ou seja, $\sin k<0 \wedge \cos k=0 \Leftrightarrow$
$\Leftrightarrow k=\frac{3\pi}{2}+2k\pi,\ k\in\mathbb{Z}$

1.6. z_6 é um número real se e só se $\text{Im}(z_6)=0$, ou seja,
$2\cos^2 k-2\sin^2 k-1=0 \Leftrightarrow$
$\Leftrightarrow 2(\cos^2 k-\sin^2 k)-1=0 \Leftrightarrow$
$\Leftrightarrow 2\cos(2k)-1=0 \Leftrightarrow$
$\Leftrightarrow \cos(2k)=\cos\left(\frac{\pi}{3}\right) \Leftrightarrow$
$\Leftrightarrow 2k=\frac{\pi}{3}+2n\pi \vee 2k=-\frac{\pi}{3}+2n\pi,\ n\in\mathbb{Z} \Leftrightarrow$
$\Leftrightarrow k=\frac{\pi}{6}+n\pi \vee k=-\frac{\pi}{6}+n\pi,\ n\in\mathbb{Z}$

2.1. $z^2+8=0 \Leftrightarrow z^2-8 \Leftrightarrow z=-\sqrt{8} \vee z=\sqrt{-8} \Leftrightarrow$
$\Leftrightarrow z=-\sqrt{8}\text{i} \vee z=\sqrt{8}\text{i} \Leftrightarrow z=-2\sqrt{2}\text{i} \vee z=2\sqrt{2}\text{i}$

2.2. $z^2-2z+4=0 \Leftrightarrow z=\frac{2\pm\sqrt{4-4\times1\times4}}{2} \Leftrightarrow z=\frac{2\pm\sqrt{-12}}{2} \Leftrightarrow$
$\Leftrightarrow z=\frac{2\pm\sqrt{12}\text{i}}{2} \Leftrightarrow z=\frac{2+2\sqrt{3}\text{i}}{2} \vee x=\frac{2-2\sqrt{3}\text{i}}{2} \Leftrightarrow$
$\Leftrightarrow z=1+\sqrt{3}\text{i} \vee z=1-\sqrt{3}\text{i}$

2.3. $z^2+27=0 \Leftrightarrow z^2=-27 \Leftrightarrow z=-\sqrt{27} \vee z=-\sqrt{27} \Leftrightarrow$
$\Leftrightarrow z=-\sqrt{27}\text{i} \vee z=\sqrt{27}\text{i} \Leftrightarrow z=-3\sqrt{3}\text{i} \vee z=3\sqrt{3}\text{i}$

2.4. $9z^2+2z+1=0 \Leftrightarrow z=\frac{-2\pm\sqrt{4-4\times9-1}}{2\times9} \Leftrightarrow$
$\Leftrightarrow z=\frac{-2\pm\sqrt{-32}}{18} \Leftrightarrow z=\frac{-2\pm\sqrt{32}\text{i}}{18} \Leftrightarrow$
$\Leftrightarrow z=\frac{-2+4\sqrt{2}\text{i}}{18} \vee z=\frac{-2-4\sqrt{2}\text{i}}{18} \Leftrightarrow$
$\Leftrightarrow z=\frac{-1+2\sqrt{2}\text{i}}{9} \vee z=\frac{-1-2\sqrt{2}\text{i}}{9}$

2.5. $z^2\left[(z-2)^2+9\right]=0 \Leftrightarrow z^2=0 \vee (z-2)^2+9=0 \Leftrightarrow$
$\Leftrightarrow z=0 \vee (z-2)^2=-9 \Leftrightarrow$
$\Leftrightarrow z=0 \vee z-2=-\sqrt{-9} \vee z-2=\sqrt{-9} \Leftrightarrow$
$\Leftrightarrow z=0 \vee z-2=-\sqrt{9}\text{i} \vee z-2=\sqrt{9}\text{i} \Leftrightarrow$
$\Leftrightarrow z=0 \vee z=2-3\text{i} \vee z=2+3\text{i}$

2.6. $z^3+5z=0 \Leftrightarrow z(z^2+5)=0 \Leftrightarrow z=0 \vee z^2+5=0 \Leftrightarrow$
$\Leftrightarrow z=0 \vee z^2=-5 \Leftrightarrow z=0 \vee z=-\sqrt{-5} \vee z=\sqrt{-5} \Leftrightarrow$
$\Leftrightarrow z=0 \vee z=-\sqrt{5}\text{i} \vee z=\sqrt{5}\text{i}$

2.7. $(z-5)^2+1=0 \Leftrightarrow (z-5)^2=-1 \Leftrightarrow$
$\Leftrightarrow z-5=-\sqrt{-1} \vee z-5=\sqrt{-1} \Leftrightarrow$
$\Leftrightarrow z-5=-\text{i} \vee z-5=\text{i} \Leftrightarrow z=5-\text{i} \vee z=5+\text{i}$

2.8. $z^3-z^2+z=0 \Leftrightarrow z(z^2-z+1)=0 \Leftrightarrow z=0 \vee z^2-z+1=0 \Leftrightarrow$
$\Leftrightarrow z=0 \vee z=\frac{1\pm\sqrt{1-4\times1\times1}}{2\times1} \Leftrightarrow z=0 \vee z=\frac{1\pm\sqrt{-3}}{2} \Leftrightarrow$
$\Leftrightarrow z=0 \vee z=\frac{1\pm\sqrt{3}\text{i}}{2} \Leftrightarrow$
$\Leftrightarrow z=0 \vee z=\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i} \vee z=\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}$

3.1. $(1-2a)-b\text{i}+3\text{i}=4-2\text{i} \Leftrightarrow$
$\Leftrightarrow (1-2a)+(3-b)\text{i}=4-2\text{i} \Leftrightarrow$
$\Leftrightarrow 1-2a=4 \wedge 3-b=-2 \Leftrightarrow 1-4=2a \wedge 3+2=b \Leftrightarrow$
$\Leftrightarrow a=-\frac{3}{2} \wedge b=5$
Portanto, $a=-\frac{3}{2}$ e $b=5$.

3.2. $a\text{i}-4+3\text{i}+2b=-\text{i} \Leftrightarrow (-4+2b)+(a+3)\text{i}=0-\text{i} \Leftrightarrow$
$\Leftrightarrow -4+2b=0 \wedge a+3=-1 \Leftrightarrow b=2 \wedge a=-4$
Portanto $a=-4$ e $b=2$.

3.3. $\text{Re}(-a+2\text{i}-3)=\text{Im}((9+a)\text{i}+4b) \Leftrightarrow$
$\Leftrightarrow \text{Re}\left[(-a-3)+2\text{i}\right]=\text{Im}\left[4b+(9+a)\text{i}\right] \Leftrightarrow$
$\Leftrightarrow -a-3=9+a \wedge b\in\mathbb{R} \Leftrightarrow$
$\Leftrightarrow -2a=12 \wedge b\in\mathbb{R} \Leftrightarrow$
$\Leftrightarrow a=-6 \wedge b\in\mathbb{R}$
Portanto, $a=-6$ e $b\in\mathbb{R}$.

3.4. $a^2\text{i}-2a+b=(b-3+a)-\text{i}+a^2\text{i}-2b\text{i} \Leftrightarrow$
$\Leftrightarrow (-2a+b)+a^2\text{i}=(b-3+a)+(-1+a^2-2b)\text{i} \Leftrightarrow$
$\Leftrightarrow -2a+b=b-3+a \wedge a^2=-1+a^2-2b \Leftrightarrow$
$\Leftrightarrow -2a=-3+a \wedge 0=-1-2b$
$\Leftrightarrow -2a=-3+a \wedge 0=-1-2b$
$\Leftrightarrow -3a=-3 \wedge 2b=-1$
$\Leftrightarrow a=1 \wedge b=-\frac{1}{2}$
Portanto, $a=1$ e $b=-\frac{1}{2}$.

4.1. $-3+2(4-i)-(3i-1)^2=-3+8-2i-(9i^2-6i+1)=$

$=5-2i-(-9-6i+1)=5-2i-(-8-6i)=$

$=5-2i+8+6i=13+4i$

4.2. $(1-\sqrt{2}i)^2+i(\sqrt{2}+i)^2=(1-2\sqrt{2}i+2i^2)+i(2+2\sqrt{2}i+i^2)=$

$=(1-2\sqrt{2}i-2)+i(2+2\sqrt{2}i-1)=$

$=(-1-2\sqrt{2}i)+i(1+2\sqrt{2}i)=-1-2\sqrt{2}i+i+2\sqrt{2}i^2=$

$=-1-2\sqrt{2}i+i-2\sqrt{2}=(-1-2\sqrt{2})+(1-2\sqrt{2})i$

4.3. $(i^2+4i)3i-2(i+2)^2=(-1+4i)3i-2(i^2+4i+4)=$

$=-3i+12i^2-2(-1+4i+4)=-3i-12-2(3+4i)=$

$=-3i-12-6-8i=-18-11i$

4.4. $(\sqrt{3}-i)^3-\sqrt{3}(-1+2i)^2=$

$=(\sqrt{3}-i)^2(\sqrt{3}-i)-\sqrt{3}(1-4i+4i^2)=$

$=(3-2\sqrt{3}i+i^2)(\sqrt{3}-i)-\sqrt{3}(1-4i-4)=$

$=(3-2\sqrt{3}i-1)(\sqrt{3}-i)-\sqrt{3}(-3-4i)=$

$=(2-2\sqrt{3}i)(\sqrt{3}-i)+3\sqrt{3}+4\sqrt{3}i=$

$=2\sqrt{3}-2i-6i+2\sqrt{3}i^2+3\sqrt{3}+4\sqrt{3}i=$

$=5\sqrt{3}-8i-2\sqrt{3}+4\sqrt{3}i=3\sqrt{3}+(4\sqrt{3}-8)i$

4.5. $2(\sqrt{2}i-i)^2-(i-2)^4=2(2i^2-2\sqrt{2}i^2+i^2)-((i-2)^2)^2=$

$=2(-2+2\sqrt{2}-1)-[(i^2-4i+4)]^2=$

$=2(-3+2\sqrt{2})-[(-1-4i+4)]^2=$

$=-6+4\sqrt{2}-(3-4i)^2=-6+4\sqrt{2}-(9-24i+16i^2)=$

$=-6+4\sqrt{2}-(9-24i-16)=$

$=-6+4\sqrt{2}-(-7-24i)=-6+4\sqrt{2}+7+24i=$

$=(1+4\sqrt{2})+24i$

4.6. $(\cos x+i\sin x)^2+(1+i)^2=$

$=(\cos^2 x+2i\cos x\sin x+i^2\sin^2 x)+(1+2i+i^2)=$

$=(\cos^2 x+2i\cos x\sin x-\sin^2 x)+(1+2i-1)=$

$=(\cos^2 x-\sin^2 x+2i\cos x\sin x)+2i=$

$=\cos(2x)+2i\cos x\sin x+2i=$

$=\cos(2x)+(2\cos x\sin x+2)i=$

$=\cos(2x)+(\sin(2x)+2)i$

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5.1. $-i(2-3i)^2+\frac{1+2i}{i}=$

$=-i(4-12i+9i^2)+\frac{(1+2i)(-i)}{i(-i)}=$

$=-i(4-12i-9)+\frac{-i-2i^2}{-i^2}=$

$=-i(-5-12i)+\frac{-i+2}{1}=5i+12i^2-i+2=$

$=5i-12-i+2=-10+4i$

Assim, a parte real é -10 e a parte imaginária é $4i$.

5.2. $\frac{2+19i}{2i+i^2}-(1+2i)^{-1}=\frac{5+19i}{2i-1}-\frac{1}{1+2i}=$

$=\frac{(5+19i)(-1-2i)}{(-1+2i)(-1-2i)}-\frac{1-2i}{(1+2i)(1-2i)}=$

$=\frac{-5-10i-19i-38i^2}{(-1)^2-(2i)^2}-\frac{1-2i}{1^2-(2i)^2}=$

$=\frac{-5-29i+38}{1-4i^2}-\frac{1-2i}{1-4i^2}=$

$=\frac{33-29i}{5}-\frac{1-2i}{5}=\frac{33-29i-1+2i}{5}=$

$=\frac{32-27i}{5}=\frac{32}{5}-\frac{27}{5}i$

Portanto, a parte real é $\frac{32}{5}$ e a parte imaginária é $-\frac{27}{5}i$.

5.3. $2-2i-\frac{i}{2-2i}+\left(\frac{1}{i}\right)^5=2-2i-\frac{i(2+2i)}{(2-2i)(2+2i)}+\frac{1}{i^5}=$

$=2-2i-\frac{2i+2i^2}{2^2-(2i)^2}+\frac{1}{i^4\times i}=2-2i-\frac{2i-2}{4-4i^2}+\frac{1}{i}=$

$=2-2i-\frac{2i-2}{8}+\frac{-i}{i(-i)}=2-2i-\frac{i-1}{4}+\frac{-i}{1}=$

$=\frac{8-8i-i+1-4i}{4}=\frac{9-13i}{4}=\frac{9}{4}-\frac{13}{4}i$

Portanto, a parte real é $\frac{9}{4}$ e a parte imaginária é $-\frac{13}{4}i$.

5.4. $i^{73}+\frac{5-10i}{1+2i}+(\sqrt{3}i)^2=i^{4\times18+1}+\frac{(5-10i)(1-2i)}{(1+2i)(1-2i)}+3i^2=$

$=i+\frac{5-10i-10i+20i^2}{1^2-(2i)^2}-3=$

$=i+\frac{5-10i-20}{1-4i^2}-3=i+\frac{-15-20i}{5}-3=$

$=i-3-4i-3=-6-3i$

Portanto, a parte real é –6 e a parte imaginária é $-3i$.

6.1. $\frac{1-i}{1+i}-\frac{\sqrt{2}+i}{2i}=\frac{(1-i)^2}{(1+i)(1-i)}-\frac{(\sqrt{2}+i)(-i)}{2i(-i)}=$

$=\frac{1-2i+i^2}{1-i^2}-\frac{-\sqrt{2}i-i^2}{-2i^2}=$

$=\frac{-2i}{2}-\frac{-\sqrt{2}i+1}{2}=-i+\frac{\sqrt{2}}{2}i-\frac{1}{2}=-\frac{1}{2}+\left(-1+\frac{\sqrt{2}}{2}\right)i$

6.2. $\frac{1}{4+i}-\frac{1+2i}{i}=\frac{4-i}{(4+i)(4-i)}-\frac{(1+2i)(-i)}{i(-i)}=$

$=\frac{4-i}{4^2-i^2}-\frac{-i-2i^2}{-i^2}=\frac{4-i}{17}-\frac{-i+2}{1}=\frac{4-i+17i-34}{17}=$

$=\frac{-30+16i}{17}=-\frac{30}{17}+\frac{16}{17}i$

6.3. $\frac{2}{(3+i)^2}-i^{-7}=\frac{2}{9+6i+i^2}-\frac{1}{i^7}=\frac{2}{8+6i}-\frac{1}{i^4\times i^3}=$

$=\frac{2(8-6i)}{(8+6i)(8-6i)}-\frac{1}{i^3}=\frac{16-12i}{8^2-(6i)^2}-\frac{1}{-i}=$

$=\frac{16-12i}{64-36i^2}-\frac{i}{-i^2}=$

$=\frac{16-12i}{100}-\frac{i}{1}=\frac{4-3i}{25}-i=\frac{4-3i-25i}{25}=$

$=\frac{4}{25}-\frac{28}{25}i$

6.4. $\frac{2+i}{(2i-1)^2}-\frac{2-i}{(2i+1)^2}=\frac{2+i}{4i^2-4i+1}-\frac{2-i}{4i^2+4i+1}=$

$=\frac{2+i}{-3-4i}-\frac{2-i}{-3+4i}=$

$=\frac{(2+i)(-3+4i)}{(-3-4i)(-3+4i)}-\frac{(2-i)(-3-4i)}{(-3+4i)(-3-4i)}=$

$=\frac{-6+8i-3i+4i^2}{(-3)^2-(4i)^2}-\frac{-6-8i+3i+4i^2}{(-3)^2-(4i)^2}=$

$=\frac{-10+5i}{9-16i^2}-\frac{-10-5i}{9-16i^2}=\frac{-10+5i}{25}-\frac{-10-5i}{25}=$

$=\frac{-10+5i+10+5i}{25}=$

$=\frac{0+10i}{25}=\frac{2}{5}i$

6.5. $\frac{1-3i}{1+2i}-(i^3-1)^2=\frac{(1-3i)(1-2i)}{(1+2i)(1-2i)}-(-i-1)^2=$

$=\frac{1-2i-3i+6i^2}{1^2-(2i)^2}-(i^2+2i+1)=\frac{-5-5i}{1-4i^2}-(-1+2i+1)=$

$=\frac{-5-5i}{5}-2i=-1-i-2i=-1-3i$

6.6. $\frac{i^{40n+1}+3-2i}{3i-1}+(1+2i)(3i^3)=$

$=\frac{i+3-2i}{3i-1}+(1+2i)(-3i)=\frac{3-i}{-1+3i}+(-3i-6i^2)=$

$=\frac{(3-i)(-1-3i)}{(-1+3i)(-1-3i)}+(-3i+6)=$

$=\frac{-3-9i+i+3i^2}{(-1)^2-(3i)^2}-3i+6=$

$=\frac{-6-8i}{1-9i^2}-3i+6=\frac{-6-8i}{10}-3i+6=$

$=\frac{-3-4i}{5}-3i+6=$

$=\frac{-3-4i-15i+30}{5}=\frac{27-19i}{5}=\frac{27}{5}-\frac{19}{5}i$

7.1. $zi=-\frac{2}{3-i}\Leftrightarrow z=\frac{2}{3i-i^2}\Leftrightarrow z=\frac{-2}{3i+1}\Leftrightarrow z=\frac{-2(-3i+1)}{3i+1(-3i+1)}\Leftrightarrow$

$\Leftrightarrow z=\frac{6i-2}{1-(3i)^2}\Leftrightarrow z=\frac{6i-2}{10}\Leftrightarrow z=-\frac{1}{5}+\frac{3}{5}i$

A solução da equação é $z=-\frac{1}{5}+\frac{3}{5}i$.

7.2. $\frac{2z+1}{i+1}=2z-i\Leftrightarrow$

$\Leftrightarrow 2z+1=(2z-i)(i+1)\Leftrightarrow$

$\Leftrightarrow 2z+1=2iz+2z-i^2-i\Leftrightarrow$

$\Leftrightarrow 2iz+1-i=1\Leftrightarrow$

$\Leftrightarrow z=\frac{i}{2i}\Leftrightarrow z=\frac{1}{2}$

A solução da equação é $z=\frac{1}{2}$

7.3. $i^{37}+z=\frac{1}{1+i}-zi\Leftrightarrow i^{4\times 9+2}+z=\frac{1-i}{(1+i)(1-i)}-zi\Leftrightarrow$

$\Leftrightarrow i^2+z=\frac{1-i}{1^2-i^2}-zi\Leftrightarrow -1+z=\frac{1-i}{2}-zi\Leftrightarrow$

$\Leftrightarrow -2+2z=1-i-2zi\Leftrightarrow$

$\Leftrightarrow 2z+2zi=1-i+2\Leftrightarrow z(2+2i)=3-i\Leftrightarrow$

$\Leftrightarrow z=\frac{3-i}{2+2i}\Leftrightarrow z=\frac{(3-i)(2-2i)}{(2+2i)(2-2i)}\Leftrightarrow z=\frac{-6-6i-2i-2}{4+4}\Leftrightarrow$

$\Leftrightarrow z=\frac{4-8i}{8}\Leftrightarrow z=\frac{1}{2}-i$

A solução da equação é $z=\frac{1}{2}-i$.

7.4. $(1+i)^{-3}=zi+z\Leftrightarrow\frac{1}{(1+i)^3}=z(1+i)\Leftrightarrow\frac{1}{(1+i)^4}=z\Leftrightarrow$

$\Leftrightarrow\frac{1}{\left[(1+i)^2\right]^2}=z\Leftrightarrow\frac{1}{\left[(1+2i+i^2)\right]^2}=z\Leftrightarrow\frac{1}{(2i)^2}=z\Leftrightarrow$

$\Leftrightarrow\frac{1}{4i^2}=z\Leftrightarrow z=-\frac{1}{4}$

A solução de equação é $z=-\frac{1}{4}$.

7.5. $(2-i)z=\frac{-z+3}{i^{12n+3}}\Leftrightarrow(2-i)z=\frac{-z+3}{i^3}\Leftrightarrow(2-i)z=\frac{-z+3}{-i}\Leftrightarrow$

$\Leftrightarrow(-2i+i^2)z=-z+3\Leftrightarrow(-2i-1)z=-z+3\Leftrightarrow$

$\Leftrightarrow(-2i-1)z+z=3\Leftrightarrow(-2i-1+1)z=3\Leftrightarrow(-2i)z=3\Leftrightarrow$

$\Leftrightarrow z=\frac{3}{-2i}\Leftrightarrow z=\frac{3i}{(-2i)i}\Leftrightarrow z=\frac{3i}{-2i^2}\Leftrightarrow z=\frac{3i}{2}\Leftrightarrow$

$\Leftrightarrow z=\frac{3}{2}i$

A solução da equação é $z=\frac{3}{2}i$.

7.6. $z-2iz=\frac{2-z}{3i+1}\Leftrightarrow(z-2iz)(3i+1)=2-z\Leftrightarrow$

$\Leftrightarrow 3zi+z-6i^2z-2iz=2-z\Leftrightarrow$

$\Leftrightarrow 3zi+z+6z-2iz+z=2\Leftrightarrow$

$\Leftrightarrow 8z+zi=2\Leftrightarrow z(8+i)=2\Leftrightarrow z=\frac{2}{8+i}\Leftrightarrow$

$\Leftrightarrow z=\frac{2(8-i)}{(8+i)(8-i)}\Leftrightarrow$

$\Leftrightarrow z=\frac{16-2i}{8^2-i^2}\Leftrightarrow z=\frac{16-2i}{64+1}\Leftrightarrow z=\frac{16}{65}-\frac{2}{65}i$

As soluções da equação é $z=\frac{16}{65}-\frac{2}{65}i$.

8.1. Substituindo na equação $1+2\mathrm{i}=3\mathrm{i}z$, a variável z por $\frac{2}{3}-\frac{1}{3}\mathrm{i}$, vem:

$1+2\mathrm{i}=3\mathrm{i}\left(\frac{2}{3}-\frac{1}{3}\mathrm{i}\right) \Leftrightarrow 1+2\mathrm{i}=2\mathrm{i}-\mathrm{i}^2 \Leftrightarrow 1+2\mathrm{i}=1+2\mathrm{i}$, que é uma igualdade verdadeira.

Portanto, $\frac{2}{3}-\frac{1}{3}\mathrm{i}$, efetivamente, solução da equação $1+2\mathrm{i}=3\mathrm{i}z$.

8.2. Tem-se que:

$$w=\frac{(1-3\mathrm{i})(2+\mathrm{i})}{1+2\mathrm{i}^{4n+1}}-\frac{3}{\mathrm{i}}=$$
$$=\frac{2+\mathrm{i}-6\mathrm{i}-3\mathrm{i}^2}{1+2\mathrm{i}}-\frac{3\mathrm{i}}{\mathrm{i}^2}=$$
$$=\frac{5-5\mathrm{i}}{1+2\mathrm{i}}-\frac{3\mathrm{i}}{-1}=$$
$$=\frac{(5-5\mathrm{i})(1-2\mathrm{i})}{(1+2\mathrm{i})(1-2\mathrm{i})}+3\mathrm{i}=\frac{5-10\mathrm{i}-5\mathrm{i}+10\mathrm{i}^2}{1^2-(2\mathrm{i})^2}+3\mathrm{i}=$$
$$=\frac{-5-15\mathrm{i}}{5}+3\mathrm{i} \Leftrightarrow w=-1-3\mathrm{i}+3\mathrm{i} \Leftrightarrow w=-1$$

Como $\mathrm{Im}(w)=0$, w é um número real.

8.3. Substituindo na equação $z^3-z(3z-4)=0$, a variável z por $\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}$, vem:

$$\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)^3-\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)\times\left(3\times\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)-4\right)=0 \Leftrightarrow$$
$$\Leftrightarrow\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)^2\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)-\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)\left(\frac{9}{2}-\frac{3\sqrt{7}}{2}\mathrm{i}-4\right)=0\Leftrightarrow$$
$$\Leftrightarrow\left(\frac{9}{4}-\frac{3\sqrt{7}}{2}\mathrm{i}+\frac{7}{4}\mathrm{i}^2\right)\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)-\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)\left(\frac{1}{2}-\frac{3\sqrt{7}}{2}\mathrm{i}\right)=0\Leftrightarrow$$
$$\Leftrightarrow\left(\frac{1}{2}-\frac{3\sqrt{7}}{2}\mathrm{i}\right)\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)-\left(\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}\right)\left(\frac{1}{2}-\frac{3\sqrt{7}}{2}\mathrm{i}\right)=0\Leftrightarrow$$

$\Leftrightarrow 0=0$, que é uma igualdade verdadeira.

Portanto, $\frac{3}{2}-\frac{\sqrt{7}}{2}\mathrm{i}$ é, efetivamente, solução da equação $z^3-z(3z-4)=0$

8.4. Tem-se que:

$$z=\frac{(4-3\mathrm{i})^2}{\mathrm{i}}-\left(2\sqrt{6}\,\mathrm{i}\right)^2=\frac{16-24\mathrm{i}+9\mathrm{i}^2}{\mathrm{i}}-24\mathrm{i}^2=$$
$$=\frac{7-24\mathrm{i}}{\mathrm{i}}+24=\frac{7}{\mathrm{i}}-\frac{24\mathrm{i}}{\mathrm{i}}+24=\frac{7}{\mathrm{i}}-24+24$$
$$=\frac{7}{\mathrm{i}}=\frac{7(-\mathrm{i})}{-\mathrm{i}^2}=\frac{-7\mathrm{i}}{1}=-7\mathrm{i}$$

Como $\mathrm{Re}(z)=0 \wedge \mathrm{Im}(z)\neq 0$, então, z é imaginário puro.

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9.1. $z_1=4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$

9.2. $z_2=3\mathrm{e}^{\mathrm{i}\pi}$

9.3. $|z_3|=\sqrt{1^2+(-1)^2}=\sqrt{2}$

Sendo $\mathrm{Arg}(z_3)=\theta$, tem-se $\tan\theta=\frac{-1}{1} \wedge \theta\in 4.^{\text{o}}\mathrm{Q}$, então, $\theta=-\frac{\pi}{4}$.

Portanto, $z_3=\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$.

9.4. $|z_4|=\sqrt{\sqrt{3}^2+(-1)^2}=2$

Sendo $\mathrm{Arg}(z_4)=\theta$, tem-se $\tan\theta=\frac{-1}{\sqrt{3}} \wedge \theta\in 4^{\text{o}}\mathrm{Q}$, ou seja, $\tan\theta=-\frac{\sqrt{3}}{3} \wedge \theta\in 4.^{\text{o}}\mathrm{Q}$, então, $\theta=-\frac{\pi}{6}$.

Portanto, $z_4=2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$.

9.5. $|z_5|=\sqrt{(-2)^2+\left(2\sqrt{3}\right)^2}=4$

Sendo $\mathrm{Arg}(z_5)=\theta$, tem-se $\tan\theta=\frac{2\sqrt{3}}{-2} \wedge \theta\in 2^{\text{o}}\mathrm{Q}$, ou seja, $\tan\theta=-\sqrt{3} \wedge \theta\in 2.^{\text{o}}\mathrm{Q}$, então, $\theta=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$.

Portanto, $z_5=4\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$.

9.6. $|z_6|=\sqrt{(-4)^2+(-4)^2}=\sqrt{32}=4\sqrt{2}$

Sendo $\mathrm{Arg}(z_6)=\theta$, tem-se $\tan\theta=\frac{-4}{-4} \wedge \theta\in 3^{\text{o}}\mathrm{Q}$, ou seja, $\tan\theta=1 \wedge \theta\in 3.^{\text{o}}\mathrm{Q}$, então, $\theta=-\pi+\frac{\pi}{4}=-\frac{3\pi}{4}$.

Portanto, $z_6=4\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}$.

9.7. $|z_7|=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$

Sendo $\mathrm{Arg}(z_7)=\theta$, tem-se $\tan\theta=\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \wedge \theta\in 4^{\text{o}}\mathrm{Q}$, ou seja, $\tan\theta=-\sqrt{3} \wedge \theta\in 4.^{\text{o}}\mathrm{Q}$, então, $\theta=-\frac{\pi}{3}$.

Portanto, $z_7=\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$.

9.8.
$$z_8=\frac{(3+\mathrm{i})^2-4+2\mathrm{i}}{2-\mathrm{i}}=\frac{9+6\mathrm{i}+\mathrm{i}^2-4+2\mathrm{i}}{2-\mathrm{i}}=$$
$$=\frac{9+6\mathrm{i}-1-4+2\mathrm{i}}{2-\mathrm{i}}=\frac{4+8\mathrm{i}}{2-\mathrm{i}}=$$
$$=\frac{(4+8\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})}=$$
$$=\frac{8+4\mathrm{i}+16\mathrm{i}+8\mathrm{i}^2}{2^2-\mathrm{i}^2}=\frac{20\mathrm{i}}{5}=4\mathrm{i}$$

Portanto, $z_8=4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

9.9. $z_9 = 2i(1+i)^{-3} \Leftrightarrow z_9 = \frac{2i}{(1+i)^3} \Leftrightarrow z_9 = \frac{2i}{(1+i)^2(1+i)} \Leftrightarrow$

$\Leftrightarrow z_9 = \frac{2i}{(1+2i+i^2)(1+i)} \Leftrightarrow z_9 = \frac{2i}{2i(1+i)} \Leftrightarrow z_9 = \frac{1}{1+i} \Leftrightarrow$

$\Leftrightarrow z_9 = \frac{1-i}{(1+i)(1-i)} \Leftrightarrow z_9 = \frac{1-i}{1^2-i^2} \Leftrightarrow$

$\Leftrightarrow z_9 = \frac{1-i}{2} \Leftrightarrow z_9 = \frac{1}{2} - \frac{1}{2}i \Leftrightarrow$

$|z_9| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$

Sendo $\operatorname{Arg}(z_9) = \theta$, tem-se $\tan\theta = \frac{-\frac{1}{2}}{\frac{1}{2}} \wedge \theta \in 4.^\text{o}\text{Q}$, ou seja,

$\tan\theta = -1 \wedge \theta \in 4.^\text{o}\text{Q}$, então $\theta = -\frac{\pi}{4}$.

Portanto, $z_9 = \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}}$.

10.1. $(1+2i)^3 + 13 =$

$= (1+2i)^2(1+2i) + 13 =$

$= (1+4i+4i^2)(1+2i) + 13 =$

$= (-3+4i)(1+2i) + 13 =$

$= -3 - 6i + 4i + 8i^2 + 13 =$

$= -11 - 2i + 13 =$

$= 2 - 2i$

Seja $w_1 = 2 - 2i$, então:

$|w_1| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

Sendo $\operatorname{Arg}(w_1) = \theta$, tem-se $\tan\theta = \frac{-2}{2} \wedge \theta \in 4.^\text{o}\text{Q}$, portanto,

$\theta = -\frac{\pi}{4}$.

Assim, $(1+2i)^3 + 13 = 2\sqrt{2}\, e^{-i\frac{\pi}{4}}$.

O módulo é $2\sqrt{2}$ e um argumento é, por exemplo, $-\frac{\pi}{4}$.

10.2. $2 - 2\sqrt{3}i^{21} + \frac{1}{2}\left(8e^{-i\frac{5\pi}{3}}\right) =$

$= 2 - 2\sqrt{3}i^{4\times5+1} + 4e^{-i\frac{5\pi}{3}} =$

$= 2 - 2\sqrt{3}i + 4\left(\cos\left(-\frac{5\pi}{3}\right) + i\sin\left(-\frac{5\pi}{3}\right)\right) =$

$= 2 - 2\sqrt{3}i + 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$

$= 2 - 2\sqrt{3}i + 2 + 2\sqrt{3}i = 4$

Então $2 - 2\sqrt{3}i^{21} + \frac{1}{2}\left(8e^{-i\frac{5\pi}{3}}\right) = 4e^{i0}$.

O módulo é 4 e um argumento é, por exemplo, 0.

10.3. $\frac{(1-\sqrt{3}i)+(i-\sqrt{3})}{\sqrt{2}-\sqrt{6}} = \frac{(1-\sqrt{3})+(1-\sqrt{3})i}{\sqrt{2}(1-\sqrt{3})} =$

$= \frac{(1-\sqrt{3})(1+i)}{\sqrt{2}(1-\sqrt{3})} = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

Seja $w_3 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, então:

$|w_3| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4}+\frac{2}{4}} = 1$

Sendo $\operatorname{Arg}(w_3) = \theta$, tem-se $\tan\theta = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \wedge \theta \in 1^\text{o}\text{Q}$, ou seja,

$\tan\theta = 1 \wedge \theta \in 1^\text{o}\text{Q}$, então, $\theta = \frac{\pi}{4}$.

Portanto, $\frac{(1-\sqrt{3}i)+(i-\sqrt{3})}{\sqrt{2}-\sqrt{6}} = e^{\frac{\pi}{4}i}$.

O módulo é 1 e um argumento é, por exemplo, $\frac{\pi}{4}$.

10.4. $\frac{4i^{120}}{-1+i^3\sqrt{3}} = \frac{4i^{4\times30}}{-1-i\sqrt{3}} = \frac{4\times1}{-1-i\sqrt{3}} =$

$= \frac{4}{-1-i\sqrt{3}} = \frac{-4}{1+i\sqrt{3}} =$

$= \frac{-4(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} =$

$= \frac{-4+4\sqrt{3}i}{1^2-(\sqrt{3}i)^2} =$

$= \frac{-4+4\sqrt{3}i}{4} =$

$= -1+\sqrt{3}i$

Seja, $w_4 = -1 + \sqrt{3}i$, então:

$|w_4| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

Sendo $\operatorname{Arg}(w_4) = \theta$, tem-se $\tan(\theta) = \frac{\sqrt{3}}{-1} \wedge \theta \in 2^\text{o}Q$, ou seja, $\tan\theta = -\sqrt{3} \wedge \theta \in 2^\text{o}Q$, então $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

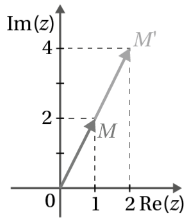
Portanto, $1+\sqrt{3}i = 2e^{i\frac{2\pi}{3}}$, isto é, $\frac{4i^{120}}{-1+i^3\sqrt{3}} = 2e^{i\frac{2\pi}{3}}$.

O módulo é 2 e um argumento é, por exemplo, $\frac{2\pi}{3}$.

11.1. $f(z) = 2z = 2(1+2i) = 2+4i$.

A imagem, pela função f, do afixo $M(1,2)$ de z do plano complexo é o afixo $M'(2,4)$.

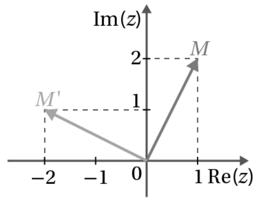
Então a imagem geométrica de $2z$ obtém-se da imagem geométrica de z por uma homotetia de centro na origem e razão 2.



11.2. $f(z) = z\,\mathrm{i} = (1+2\mathrm{i})\mathrm{i} = -2-\mathrm{i}$

A imagem, pela função f, do afixo $M(1, 2)$ do plano complexo é o afixo $M'(-2, -1)$.

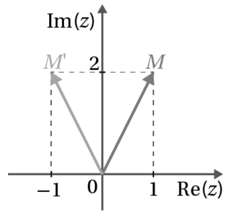
Então, a imagem geométrica de $\mathrm{i}z$ obtém-se da imagem geométrica de z por uma rotação com centro na origem e amplitude $\frac{\pi}{2}$.



11.3. $f(z) = -\bar{z} = -(1-2\mathrm{i}) = -1+2\mathrm{i}$

A imagem, pela função f, do afixo $M(1, 2)$ do plano complexo é o afixo $M'(-1, 2)$.

Então, a imagem geométrica de $-\bar{z}$ obtém-se da imagem geométrica de z por uma reflexão relativamente ao eixo imaginário.



12.1. $z_1 \times z_2 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}} \times \left(-3-\sqrt{27}\,\mathrm{i}\right)$

Tem-se que:

$$|z_2| = \sqrt{(-3)^2 + \left(-\sqrt{27}\right)^2} = \sqrt{9+27} = \sqrt{36} = 6$$

Sendo $\mathrm{Arg}(z_2) = \theta$, vem que $\tan\theta = \frac{-\sqrt{27}}{-3} \wedge \theta \in 3.^{\text{o}}\mathrm{Q}$, ou

seja, $\tan\theta = \frac{-3\sqrt{3}}{-3} \wedge \theta \in 3^{\text{o}}\mathrm{Q}$, isto é, $\tan\theta \wedge \theta \in 3.^{\text{o}}\mathrm{Q}$,

então $\theta = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$.

Portanto, $z_2 = 6\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$.

$$z_1 \times z_2 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}} \times \left(-3-\sqrt{27}\,\mathrm{i}\right) =$$

$$= 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}} \times 6\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} =$$

$$= 2\times 6\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}-\frac{2\pi}{3}\right)} =$$

$$= 12\,\mathrm{e}^{-\mathrm{i}\pi} =$$

$$= 12\,\mathrm{e}^{\mathrm{i}\pi}$$

12.2. $2z_1 \times \bar{z}_3 = 2\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right) \times \bar{z}_3$.

$z_3 = \frac{4}{\mathrm{i}} = \frac{4\mathrm{i}}{\mathrm{i}^2} = \frac{4\mathrm{i}}{-1} = -4\mathrm{i} = 4\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$, então, $\bar{z}_3 = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

$$2z_1 \times \bar{z}_3 = 2\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right) \times \bar{z}_3 =$$

$$= 4\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}} \times 4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} =$$

$$= 4\times 4\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}+\frac{\pi}{2}\right)} =$$

$$= 16\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$$

12.3. $-\frac{\bar{z}_2}{\frac{1}{2}z_1} = -\frac{6\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}}{\frac{1}{2}\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)} = -\frac{6\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}}{\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}} = -6\mathrm{e}^{\left(\frac{2\pi}{3}+\frac{\pi}{3}\right)\mathrm{i}} =$

$$= -6\mathrm{e}^{\mathrm{i}\pi} = 6\mathrm{e}^{\mathrm{i}(-\pi+\pi)} = 6\mathrm{e}^{\mathrm{i}0}$$

12.4. $\frac{-\bar{z}_3 \times z_1}{z_2} = \frac{-4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} \times 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}}{6\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}} = \frac{4\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\pi\right)} \times 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}}{6\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}} =$

$$= \frac{4\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}} \times 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}}{6\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}} = \frac{8\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{2}-\frac{\pi}{3}\right)}}{6\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}} = \frac{8\mathrm{e}^{\mathrm{i}\frac{7\pi}{6}}}{6\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}} =$$

$$= \frac{8}{6}\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{6}+\frac{2\pi}{3}\right)} = \frac{4}{3}\mathrm{e}^{\mathrm{i}\frac{11\pi}{6}} = \frac{4}{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$$

12.5. $-\frac{1}{3}\bar{z}_2 \times \frac{1}{\bar{z}_3} = -\frac{1}{3}\left(6\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right) \times \frac{1}{4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}} =$

$$= \frac{1}{3}\left(6\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}+\pi\right)}\right) \times \frac{1}{4\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}} = 2\mathrm{e}^{\mathrm{i}\frac{5\pi}{3}} \times \frac{1}{4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}} = \frac{2\mathrm{e}^{\mathrm{i}\frac{5\pi}{3}}}{4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}} =$$

$$= \frac{2}{4}\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{3}-\frac{\pi}{2}\right)} = \frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{7\pi}{6}} = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}$$

12.6. $z_1 + z_2 + 2\sqrt{3}\,\mathrm{i} = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}} + \left(-3-\sqrt{27}\,\mathrm{i}\right) + 2\sqrt{3}\,\mathrm{i} =$

$$= 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right) + \left(-3-3\sqrt{3}\,\mathrm{i}\right) + 2\sqrt{3}\,\mathrm{i}$$

$$= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) - 3 - \sqrt{3}\,\mathrm{i} = 1 - \sqrt{3}\,\mathrm{i} - 3 - \sqrt{3}\,\mathrm{i} =$$

$$= -2 - 2\sqrt{3}\,\mathrm{i}$$

Seja $w = -2-2\sqrt{3}\,\mathrm{i}$, então:

$$|w| = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$$

Sendo $\mathrm{Arg}(w) = \theta$, tem-se $\tan\theta = \frac{-2\sqrt{3}}{-2} \wedge \theta \in 3^{\text{o}}\mathrm{Q}$, isto é,

$\tan\theta = \sqrt{3} \wedge \theta \in 3^{\text{o}}\mathrm{Q}$, então, $\theta = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$.

Portanto, $-2-2\sqrt{3}\,\mathrm{i} = 4\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$, ou seja,

$z_1 + z_2 + 2\sqrt{3}\,\mathrm{i} = 4\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$.

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13.1. $z = 2e^{-i\frac{2\pi}{3}} = 2\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right] = 2\left(-\frac{1}{3} - \frac{\sqrt{3}}{2}i\right) =$

$= -1 - \sqrt{3}\,i$

Logo, $\bar{z} = -1+\sqrt{3}i$ e $-z = 1+\sqrt{3}i$.

13.2. $z = e^{-i\frac{\pi}{4}} - 2e^{-i\pi} = \left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right] - (-2)$

$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i + 2 = \left(\frac{\sqrt{2}}{2}+2\right) - \frac{\sqrt{2}}{2}i$

Portanto, $\bar{z} = \left(2+\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}i$ e $-z = \left(-2-\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}i$.

13.3. $z = \frac{8i\,e^{i\frac{\pi}{6}}}{e^{-i\frac{\pi}{4}}} = \frac{8i\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)} =$

$= \frac{8i\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)}{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i} = \frac{-4+4\sqrt{3}i}{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i} =$

$= \frac{\left(-4+4\sqrt{3}i\right)\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)}{\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)} =$

$= \frac{-2\sqrt{2}-2\sqrt{2}i+2\sqrt{6}i-2\sqrt{6}}{\frac{1}{6}+\frac{1}{2}} =$

$= \left(-2\sqrt{6}-2\sqrt{2}\right)+\left(2\sqrt{6}-2\sqrt{2}\right)i$

$\bar{z} = \left(-2\sqrt{6}-2\sqrt{2}\right)-\left(2\sqrt{6}-2\sqrt{2}\right)i$

$-z = \left(2\sqrt{6}+2\sqrt{2}\right)+\left(2\sqrt{2}-2\sqrt{6}\right)i$

13.4. $z = \frac{e^{-i\frac{\pi}{2}}\times\left(\sqrt{3}-i\right)}{i^7\,e^{i\frac{2\pi}{3}}} = \frac{-i\times\left(\sqrt{3}-i\right)}{-i\,e^{i\frac{2\pi}{3}}} = \frac{\sqrt{3}-i}{e^{i\frac{2\pi}{3}}} =$

$= \frac{\sqrt{3}-i}{\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)} = \frac{\sqrt{3}-i}{-\frac{1}{2}+\frac{\sqrt{3}}{2}i} =$

$= \frac{\left(\sqrt{3}-i\right)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)}{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)} =$

$= \frac{-\frac{\sqrt{3}}{2}-\frac{3}{2}i+\frac{1}{2}i-\frac{\sqrt{3}}{2}}{\frac{1}{4}+\frac{3}{4}} =$

$= \frac{-\sqrt{3}-i}{1} =$

$= -\sqrt{3}-i$

Portanto, $\bar{z} = -\sqrt{3}+i$ e $-z = \sqrt{3}+i$.

14.1. $z_1 - i = \sqrt{3}\,e^{i\frac{2\pi}{3}} - i = \sqrt{3}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) - i =$

$= \sqrt{3}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) - i = -\frac{\sqrt{3}}{2}+\frac{3}{2}i - i = -\frac{\sqrt{3}}{2}+\frac{1}{2}i$

$|z_1 - i| = \left|-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$

Sendo $\operatorname{Arg}(z_1 - i) = \theta$, tem-se: $\tan\theta = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} \wedge \theta \in 2.^{\text{o}}\text{Q}$, ou seja, $\tan\theta = -\frac{\sqrt{3}}{3} \wedge \theta \in 2.^{\text{o}}\text{Q}$, então $\theta = \frac{5\pi}{6}$.

Portanto, $z_1 - i = e^{i\frac{5\pi}{6}}$.

14.2. $\frac{z_3\times z_2 - 2}{\sqrt{3}\,i^5} = \frac{(2+2i)\left(\sqrt{2}\,e^{i\frac{\pi}{12}}\right)-2}{\sqrt{3}\,i^{4+1}} = \frac{(2+2i)\left(\sqrt{2}\,e^{i\frac{\pi}{12}}\right)-2}{\sqrt{3}\,i}$

$z_3 = 2+2i$

$|z_3| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

Sendo $\operatorname{Arg}(z_3) = \theta$, tem-se $\tan\theta = \frac{2}{2} \wedge \theta \in 1.^{\text{o}}\text{Q}$, logo $\theta = \frac{\pi}{4}$. Assim, $z_3 = 2\sqrt{2}\,e^{i\frac{\pi}{4}}$.

Voltando ao cálculo de $\frac{z_3\times z_2 - 2}{\sqrt{3}\,i^5}$, vem:

$\frac{(2+2i)\left(\sqrt{2}\,e^{i\frac{\pi}{12}}\right)-2}{\sqrt{3}\,i} = \frac{\left(2\sqrt{2}\,e^{i\frac{\pi}{4}}\right)\left(\sqrt{2}\,e^{i\frac{\pi}{12}}\right)-2}{\sqrt{3}\,i} =$

$= \frac{2\sqrt{2}\times\sqrt{2}\,e^{i\left(\frac{\pi}{4}+\frac{\pi}{12}\right)}-2}{\sqrt{3}\,i} = \frac{4e^{i\frac{\pi}{3}}-2}{\sqrt{3}\,i} = \frac{4\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}-2}{\sqrt{3}\,i} =$

$= \frac{4\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)-2}{\sqrt{3}\,i} = \frac{2+2\sqrt{3}i-2}{\sqrt{3}\,i} = \frac{2\sqrt{3}i}{\sqrt{3}\,i} = 2$

15.1. $\frac{\bar{z}_1}{\frac{\sqrt{3}}{2}+z_1} = \frac{\sqrt{3}\,e^{i\frac{2\pi}{3}}}{\frac{\sqrt{3}}{2}+\sqrt{3}\,e^{-i\frac{2\pi}{3}}} =$

$= \frac{\sqrt{3}\,e^{i\frac{2\pi}{3}}}{\frac{\sqrt{3}}{2}+\sqrt{3}\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{3\pi}{3}\right)\right)} =$

$= \frac{\sqrt{3}^{i\frac{2\pi}{3}}}{\frac{\sqrt{3}}{2}+\sqrt{3}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)} = \frac{\sqrt{3}\,e^{\frac{2\pi}{3}i}}{\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}-\frac{3}{2}i} =$

$= \frac{\sqrt{3}\,e^{i\frac{2\pi}{3}}}{-\frac{3}{2}i} = \frac{\sqrt{3}\,e^{i\frac{2\pi}{3}}}{\frac{3}{2}e^{-i\frac{\pi}{2}}} = \frac{\sqrt{3}}{\frac{3}{2}}e^{i\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)} =$

$= \frac{2\sqrt{3}}{3}e^{i\frac{7\pi}{6}} = \frac{2\sqrt{3}}{3}e^{-i\frac{5\pi}{6}}$

15.2. $w=\frac{-z_1\times z_2}{\mathrm{i}}=\frac{-\sqrt{3}\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\times\left(\frac{\sqrt{2}}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}-2^{-1}\right)}{\mathrm{i}}=$

$=\frac{\sqrt{3}\,\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}+\pi\right)}\times\left[\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)-\frac{1}{2}\right]}{\mathrm{i}}=$

$=\frac{\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\times\left[\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}\right)-\frac{1}{2}\right]}{\mathrm{i}}=$

$=\frac{\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\times\left(\frac{1}{2}+\frac{1}{2}\mathrm{i}-\frac{1}{2}\right)}{\mathrm{i}}\Leftrightarrow w=\frac{\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\times\left(\frac{1}{2}\mathrm{i}\right)}{\mathrm{i}}=$

$=\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\times\frac{1}{2}\Leftrightarrow w=\frac{\sqrt{3}}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

16.1. Tem-se $|z|=\sqrt{1+3^2}=\sqrt{10}$, pelo que $z=\sqrt{10}\,\mathrm{e}^{\mathrm{i}\theta}$.

Portanto, $-z=-\sqrt{10}\,\mathrm{e}^{\mathrm{i}\theta}=\sqrt{10}\,\mathrm{e}^{\mathrm{i}(\theta+\pi)}$, onde

$\mathrm{i}\times(-z)=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\times\sqrt{10}\,\mathrm{e}^{\mathrm{i}(\theta+\pi)}=\sqrt{10}\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\theta+\pi\right)}=\sqrt{10}\,\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{2}+\theta\right)}$

16.2. Sendo $\mathrm{Arg}(z)=\theta$, tem-se que $\tan\theta=\frac{3}{1}=3$

Recorrendo à fórmula $1+\tan^2\theta=\frac{1}{\cos^2\theta}$ e substituindo nesta $\tan\theta$ por 3, vem:

$3=\frac{\sin\theta}{\frac{\sqrt{10}}{10}}\Leftrightarrow\sin\theta=\frac{3\sqrt{10}}{10}$

Assim, tem-se:

$\cos\left(\theta+\frac{\pi}{6}\right)=\cos\theta\cos\frac{\pi}{6}-\sin\theta\frac{\pi}{6}=$

$=\frac{\sqrt{10}}{10}\times\frac{\sqrt{3}}{2}-\frac{3\sqrt{10}}{10}\times\frac{1}{2}=$

$=\frac{\sqrt{30}}{20}-\frac{3\sqrt{10}}{20}=\frac{\sqrt{30}-3\sqrt{10}}{20}$

17.1. $\mathrm{e}^{-\mathrm{i}\pi}\times(z_2)^{-3}=\mathrm{e}^{-\mathrm{i}\pi}\times(2+3\mathrm{i})^{-3}=$

$=\left[\cos(-\pi)+\mathrm{i}\sin(-\pi)\times\frac{1}{(2+3\mathrm{i})^3}\right]=$

$=-1\times\frac{1}{(2+3\mathrm{i})^3}=\frac{-1}{(2+3\mathrm{i})^3}=\frac{-1}{(2+3\mathrm{i})^2(2+3\mathrm{i})}=$

$=\frac{-1}{(4+12\mathrm{i}+9\mathrm{i}^2)(2+3\mathrm{i})}=\frac{-1}{(-5+12\mathrm{i})(2+3\mathrm{i})}=$

$=\frac{-1}{-10-15\mathrm{i}+24\mathrm{i}+36\mathrm{i}^2}=$

$=\frac{-1}{-46+9\mathrm{i}}=\frac{1}{46-9\mathrm{i}}=\frac{46+9\mathrm{i}}{46^2-(9\mathrm{i})^2}=$

$=\frac{46+9\mathrm{i}}{2116+81}=\frac{46+9\mathrm{i}}{2197}=\frac{46}{2197}+\frac{9}{2197}\mathrm{i}$

17.2. $|z_1|-\bar{z}_2\times z=\mathrm{i}z\Leftrightarrow\sqrt{\left(-\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)^2}-(2-3\mathrm{i})z=\mathrm{i}z\Leftrightarrow$

$\Leftrightarrow 1-(2-3\mathrm{i})z=\mathrm{i}z\Leftrightarrow 1=(2-3\mathrm{i})z+\mathrm{i}z\Leftrightarrow$

$\Leftrightarrow 1=(2-3\mathrm{i}+-\mathrm{i})z\Leftrightarrow 1=(2-2\mathrm{i})z\Leftrightarrow$

$\Leftrightarrow z=\frac{1}{2-2\mathrm{i}}\Leftrightarrow z=\frac{2+2\mathrm{i}}{(2-2\mathrm{i})(2+2\mathrm{i})}\Leftrightarrow$

$\Leftrightarrow z=\frac{2+2\mathrm{i}}{2^2-(2\mathrm{i})^2}\Leftrightarrow z=\frac{2+2\mathrm{i}}{4+4}\Leftrightarrow z=\frac{2+2\mathrm{i}}{8}\Leftrightarrow$

$\Leftrightarrow z=\frac{1}{4}+\frac{1}{4}\mathrm{i}$

$|z|=\sqrt{\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2}=\sqrt{\frac{1}{16}+\frac{1}{16}}=\sqrt{\frac{2}{16}}=\frac{\sqrt{2}}{4}$

Sendo θ um argumento de z tem-se

$\tan\theta=\frac{\frac{1}{4}}{\frac{1}{4}}\wedge\theta\in 1.^{\text{o}}\mathrm{Q}$, ou seja, $\tan\theta=1\wedge\theta\in 1.^{\text{o}}\mathrm{Q}$, então

$\theta=\frac{\pi}{4}$.

Portanto, a solução da equação é $z=\frac{\sqrt{2}}{4}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$.

Ficha para praticar 18

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1.1. $\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{5}}\right)^4=2^4\,\mathrm{e}^{-\mathrm{i}\frac{4\pi}{5}}=16\mathrm{e}^{-\mathrm{i}\frac{4\pi}{5}}$

1.2. Seja $z_1=\sqrt{6}-\sqrt{2}\,\mathrm{i}$

$|z_1|=\sqrt{\left(\sqrt{6}\right)^2\left(-\sqrt{2}\right)^2}=\sqrt{6+2}=\sqrt{8}=2\sqrt{2}$

Sendo $\mathrm{Arg}(z_1)=\theta$, tem-se $\tan\theta=\frac{-\sqrt{2}}{\sqrt{6}}\wedge\theta\in 4.^{\text{o}}\mathrm{Q}$, ou seja,

$\tan\theta=-\frac{\sqrt{3}}{3}\wedge\theta\in 4.^{\text{o}}\mathrm{Q}$, então, $\theta=-\frac{\pi}{6}$.

Portanto, $z_1=2\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$.

Assim, $\left(\sqrt{6}-\sqrt{2}\,\mathrm{i}\right)^6=\left(2\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\right)^6=\left(2\sqrt{2}\right)^6\mathrm{e}^{-\mathrm{i}\frac{6\pi}{6}}=512\mathrm{e}^{-\mathrm{i}\pi}$.

1.3. Seja $z_2=\frac{(3+\mathrm{i})^2}{2-\mathrm{i}}-2$.

$\frac{(3+\mathrm{i})^2}{2-\mathrm{i}}-2=\frac{9+6\mathrm{i}+\mathrm{i}^2}{2-\mathrm{i}}-2=\frac{8+6\mathrm{i}}{2-\mathrm{i}}-2=\frac{(8+6\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})}-2=$

$=\frac{16+8\mathrm{i}+12\mathrm{i}+6\mathrm{i}^2}{2^2-\mathrm{i}^2}-2=\frac{10+20\mathrm{i}}{5}-2=2+4\mathrm{i}-2=4\mathrm{i}$

Portanto, $z_2=4\mathrm{i}=4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

Assim, $\left(\frac{(3+\mathrm{i})^2}{2-\mathrm{i}}-2\right)^5=\left(4\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)=4^5\,\mathrm{e}^{\mathrm{i}\frac{5\pi}{2}}=1024\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$

1.4. Seja $z_3 = \sqrt{3} - \mathrm{i}$

$|z_3| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$

Sendo $\operatorname{Arg}|z_3| = \theta$, tem-se $\tan\theta = \frac{-1}{\sqrt{3}} \wedge \theta \in 4.^{\text{o}}\text{Q}$, ou seja,

$\tan\theta = -\frac{\sqrt{3}}{3} \wedge \theta \in 4.^{\text{o}}\text{Q}$, então $\theta = -\frac{\pi}{6}$.

Portanto, $z_3 = 2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$.

Assim, $\left(\frac{\sqrt{3}-\mathrm{i}}{\sqrt{6}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}\right)^3 = \left(\frac{2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}}{\sqrt{6}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}\right)^3 = \left[\frac{2}{\sqrt{6}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}+\frac{\pi}{4}\right)}\right]^3 =$

$= \left(\frac{2}{\sqrt{6}}\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}\right)^3 = \left(\frac{2}{\sqrt{6}}\right)^3 \mathrm{e}^{\mathrm{i}\frac{3\pi}{12}} = \frac{8}{6\sqrt{6}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} = \frac{2\sqrt{6}}{9}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$

2.1. $\left(-1+\sqrt{3}\,\mathrm{i}\right)^6$

Seja $z_1 = -1 + \sqrt{3}\,\mathrm{i}$

$|z_1| = \sqrt{(-1)^2 + \sqrt{3}^2} = 2$

Sendo $\operatorname{Arg}(z_1) = \theta$, tem-se $\tan\theta = \frac{\sqrt{3}}{-1} \wedge \theta \in 2.^{\text{o}}\text{Q}$, então,

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Portanto, $z_1 = 2\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$.

Assim:

$\left(-1+\sqrt{3}\,\mathrm{i}\right)^6 = \left(2\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^6 = \left(2\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^6 = 2^6\,\mathrm{e}^{\mathrm{i}\left(6\times\frac{2\pi}{3}\right)}$

$= 64\,\mathrm{e}^{\mathrm{i}4\pi} = 64\,\mathrm{e}^{\mathrm{i}0} = 64$

2.2. $\left(\frac{1-\mathrm{i}}{1+\mathrm{i}}\right)^8 = \left[\frac{(1-\mathrm{i})^2}{(1+\mathrm{i})(1-\mathrm{i})}\right]^8 = \left(\frac{1-2\mathrm{i}+\mathrm{i}^2}{1^2-\mathrm{i}^2}\right)^8 =$

$= \left(\frac{-2\mathrm{i}}{2}\right)^8 = (-\mathrm{i})^8 = \mathrm{i}^8 = 1$

2.3. $\left(\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}+1\right)^{10} = \left(\cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3} + 1\right)^{10} = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} + 1\right)^{10} =$

$= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)^{10}$

Seja $z_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

$|z_3| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

Sendo $\operatorname{Arg}(z_3) = \theta$, tem-se $\tan\theta = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \wedge \theta \in 4.^{\text{o}}\text{Q}$, isto é,

$\tan\theta = -\sqrt{3} \wedge \theta \in 4.^{\text{o}}\text{Q}$, então $\theta = -\frac{\pi}{3}$.

Portanto, $z_3 = \mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$.

Assim:

$\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)^{10} = \left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)^{10} = \mathrm{e}^{-\mathrm{i}\frac{10\pi}{3}} = \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} =$

$= \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

2.4. $\left(\frac{-2-\mathrm{i}}{\mathrm{i}} - 1\right)^{12} = \left(\frac{-2-\mathrm{i}-\mathrm{i}}{\mathrm{i}}\right)^{12} = \left(\frac{-2-2\mathrm{i}}{\mathrm{i}}\right) = \left(\frac{-2\mathrm{i}-2\mathrm{i}^2}{\mathrm{i}^2}\right)^{12} =$

$= \left(\frac{-2\mathrm{i}+2}{-1}\right)^{12} = (-2+2\mathrm{i})^{12}$

Seja $z_4 = -2 + 2\mathrm{i}$.

$|z_4| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$.

Sendo $\operatorname{Arg}(z_4) = \theta$, tem-se $\tan\theta = \frac{2}{-2} \wedge \theta \in 2.^{\text{o}}\text{Q}$, ou seja,

$\tan\theta = -1 \wedge \theta \in 2.^{\text{o}}\text{Q}$, então, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

Portanto, $z_4 = 2\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$.

Assim:

$(-2+2\mathrm{i})^{12} = \left(2\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\right)^{12} = \left(2\sqrt{2}\right)^{12}\mathrm{e}^{\mathrm{i}\left(12\times\frac{3\pi}{4}\right)}$

$= 262\,144\,\mathrm{e}^{\mathrm{i}9\pi} = 212\,144\,\mathrm{e}^{\mathrm{i}\pi} = -262\,144$

3.1. Os dois números têm o mesmo módulo (no caso 2) pelo que são iguais se a diferença dos seus argumentos for múltipla de 2π.

$\frac{43\pi}{35} - \frac{\pi}{35} = 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow k = \frac{42}{70},\ k \in \mathbb{Z} \Leftrightarrow k = \frac{3}{5},\ k \in \mathbb{Z}$

Como $\frac{3}{5} \notin \mathbb{Z}$, os números z_1 e z_2 são diferentes.

Por outro lado, são raízes de índice 5 do mesmo número se e somente se:

$(z_1)^5 = (z_2)^5$

$(z_1)^5 = \left(2\,\mathrm{e}^{\mathrm{i}\frac{\pi}{35}}\right)^5 = 2^5\,\mathrm{e}^{\mathrm{i}\frac{5\pi}{35}} = 32\,\mathrm{e}^{\mathrm{i}\frac{\pi}{7}}$

$(z_2)^5 = \left(2\,\mathrm{e}^{\mathrm{i}\frac{43\pi}{35}}\right)^5 = 2^5\,\mathrm{e}^{\mathrm{i}\left(5\times\frac{43\pi}{35}\right)} = 32\,\mathrm{e}^{\mathrm{i}\frac{43\pi}{7}} = 32\,\mathrm{e}^{\mathrm{i}\left(\frac{42\pi}{7}+\frac{\pi}{7}\right)} =$

$= 32\,\mathrm{e}^{\mathrm{i}\left(6\pi+\frac{\pi}{7}\right)} = 3\,\mathrm{e}^{\mathrm{i}\frac{\pi}{7}}$

Como $(z_1)^5 = (z_2)^5$, então, os dois números são, efetivamente, raízes de índice 5 do mesmo número, sendo este igual a $32\,\mathrm{e}^{\mathrm{i}\frac{\pi}{7}}$.

3.2. $(z_2)^n = \left(2\,\mathrm{e}^{\mathrm{i}\frac{43\pi}{35}}\right)^n = 2^n\,\mathrm{e}^{\mathrm{i}\frac{43\pi}{35}n}$

$2^n\,\mathrm{e}^{\mathrm{i}\frac{43\pi}{35}n}$ é um número real positivo, se e só se

$\frac{43\pi}{35}n = 2k\pi,\ k \in \mathbb{Z}$.

$\frac{43\pi}{35}n = 2k\pi, k \in \mathbb{Z} \Leftrightarrow 43\pi n = 70k\pi, k \in \mathbb{Z} \Leftrightarrow n = \frac{70}{43}k, k \in \mathbb{Z}$.

Portanto, o menor valor natural de n é obtido para $k = 43$, ou seja, é $n = 70$.

4.1. $\left(\overline{z_1} \times \frac{1}{z_1}\right)^5 = \left(2\,e^{-i\frac{\pi}{5}} \times \frac{1}{2\,e^{i\frac{\pi}{5}}}\right)^5 = \left(\frac{2\,e^{-i\frac{\pi}{5}}}{2\,e^{i\frac{\pi}{5}}}\right)^5 = \left(\frac{e^{-i\frac{\pi}{5}}}{e^{i\frac{\pi}{5}}}\right)^5 =$

$= \left(e^{-i\frac{2\pi}{5}}\right)^5 = e^{-i\times 2\pi} = e^{-i\times 0} = 1$

4.2. $z_2 \times (i z_1) = e^{i\theta} \times \left(i \times 2\,e^{i\frac{\pi}{5}}\right) =$

$= e^{i\theta} \times \left(e^{i\frac{\pi}{2}} \times 2\,e^{i\frac{\pi}{5}}\right) =$

$= e^{i\theta} \times \left(2\,e^{i\left(\frac{\pi}{2}+\frac{\pi}{5}\right)}\right) =$

$= e^{i\theta} \times 2\,e^{i\frac{7\pi}{10}} =$

$= 2\,e^{i\left(\theta + \frac{7\pi}{10}\right)}$

O número $2\,e^{i\left(\theta + \frac{7\pi}{10}\right)}$ é um imaginário puro de coeficiente positivo se e somente se

$\theta + \frac{7\pi}{10} = \frac{\pi}{2} + 2\pi, k \in \mathbb{Z} \Leftrightarrow \theta = \frac{\pi}{2} - \frac{7\pi}{10} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow \theta = -\frac{2\pi}{10} + 2k\pi, k \in \mathbb{Z} \Leftrightarrow \theta = -\frac{\pi}{5} + 2k\pi, k \in \mathbb{Z}$

Como $\theta \in \left]-\pi, \pi\right[$, vem que $\theta = -\frac{\pi}{5}$.

5.1. $w = \frac{\left(\overline{z_1}\right)^5 \times |z_1| \times (-z_1)}{\frac{1}{2} z_1} =$

$= \frac{\left(2\,e^{i\frac{\pi}{3}}\right) \times 2 \times (-1)}{\frac{1}{2}} =$

$= -4 \times 2^5\,e^{-i\frac{5\pi}{3}} =$

$= 4 \times 32\,e^{-i\left(\frac{5\pi}{3} - \pi\right)} =$

$= 128\,e^{-i\frac{2\pi}{3}}$

O argumento principal de w deve pertencer ao intervalo $\left]-\pi, \pi\right]$, portanto, este é $\frac{-2\pi}{3}$.

5.2. $(z_2)^4 \times z_1 = z\,i \Leftrightarrow \left(-\sqrt{2} - \sqrt{2}\,i\right)^4 \left(2\,e^{i\frac{\pi}{3}}\right) = z\,i$

Vamos escrever z_2 na forma trigonométrica.

$|z_2| = \sqrt{\left(-\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)} = 2$

Sendo $\text{Arg}(z_2) = \theta$, tem-se $\tan\theta = \frac{-\sqrt{2}}{-\sqrt{2}} \wedge \theta \in 3.^{\text{o}}\text{Q}$, ou seja,

$\tan\theta = 1 \wedge \theta \in 3.^{\text{o}}\text{Q}$, então, $\theta = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$.

Portanto, $z_2 = 2\,e^{-i\frac{3\pi}{4}}$.

Voltando à equação, tem-se:

$\left(-\sqrt{2} - \sqrt{2}\,i\right)^4 \left(2\,e^{i\frac{\pi}{3}}\right) = z\,i \Leftrightarrow$

$\Leftrightarrow \left(2\,e^{-i\frac{3\pi}{4}}\right)^4 \left(2\,e^{i\frac{\pi}{3}}\right) = z\,i \Leftrightarrow$

$\Leftrightarrow \left(2^4\,e^{-i\frac{3\pi}{4}\times 4}\right)\left(2\,e^{i\frac{\pi}{3}}\right) = z\,i \Leftrightarrow$

$\Leftrightarrow 16\,e^{i3\pi} \times 2\,e^{i\frac{\pi}{3}} = z\,i \Leftrightarrow$

$\Leftrightarrow 32\,e^{-i\frac{2\pi}{3}} = z\,i \Leftrightarrow$

$\Leftrightarrow 32\,e^{-i\frac{2\pi}{3}} = z\,i \Leftrightarrow$

$\Leftrightarrow \frac{32\,e^{-i\frac{2\pi}{3}}}{i} = z \Leftrightarrow$

$\Leftrightarrow \frac{32\,e^{-i\frac{2\pi}{3}}}{e^{i\frac{\pi}{2}}} = z \Leftrightarrow$

$\Leftrightarrow 32\,e^{i\left(-\frac{2\pi}{3} - \frac{\pi}{2}\right)} = z \Leftrightarrow$

$\Leftrightarrow 32\,e^{-i\frac{7\pi}{6}} = z \Leftrightarrow$

$\Leftrightarrow 32\,e^{i\frac{5\pi}{6}} = z \Leftrightarrow$

$\Leftrightarrow 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = z \Leftrightarrow$

$\Leftrightarrow 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = z \Leftrightarrow$

$\Leftrightarrow -16\sqrt{3} + 16i = z$

A solução da equação é $z = -16\sqrt{3} + 16i$.

5.3. $w = \frac{-2}{z_2 \times (z_3)^3} \Leftrightarrow w = \frac{-2}{2\,e^{-i\frac{3\pi}{4}} \times \left(2\,e^{i\theta}\right)^3} \Leftrightarrow$

$\Leftrightarrow w = \frac{-2}{2\,e^{-i\frac{3\pi}{4}} \times 8\,e^{i3\theta}} \Leftrightarrow w = \frac{-2\,e^{i\pi}}{16\,e^{i\left(-\frac{3\pi}{4} + 3\theta\right)}} \Leftrightarrow$

$\Leftrightarrow w = \frac{1}{8}e^{i\left(\pi + \frac{3\pi}{4} - 3\theta\right)} \Leftrightarrow$

$\Leftrightarrow w = \frac{1}{8}e^{i\left(\frac{7\pi}{4} - 3\theta\right)} \Leftrightarrow$

$\Leftrightarrow w = \frac{1}{8}e^{i\left(-\frac{\pi}{4} + 3\theta\right)}$

O número w é um número real se e somente se

$-\frac{\pi}{4} - 3\theta = k\pi, k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow -3\theta = \frac{\pi}{4} + k\pi, k \in \mathbb{Z} \Leftrightarrow$

$\Leftrightarrow \theta = -\frac{\pi}{12} - \frac{k\pi}{3}, k \in \mathbb{Z}$

6.1. $|z_1| = \sqrt{1^2+1^2} = \sqrt{2}$

Sendo $\operatorname{Arg}(z_1)=\theta$, tem-se $\tan\theta = \frac{1}{1} \wedge \theta \in 1.^{\text{o}}\text{Q}$, então,

$\theta = \frac{\pi}{4}$. Portanto, $z_1 = \sqrt{2}\,e^{i\frac{\pi}{4}}$. Assim, vem:

$$w = \frac{(z_1)^{10}}{(\overline{z_1})^6} = \frac{\left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^{10}}{\left(\sqrt{2}\,e^{-i\frac{\pi}{4}}\right)^6} = \frac{\left(\sqrt{2}\right)^{10}}{\left(\sqrt{2}\right)^6} e^{i\left(\frac{10\pi}{4}+\frac{6\pi}{4}\right)} =$$

$$= \left(\sqrt{2}\right)^4 e^{i4\pi} = 4e^{i0} = 4 \in \mathbb{R}$$

Logo, w é um número real.

6.2. Sabemos que $z_1 = \sqrt{2}\,e^{i\frac{\pi}{4}}$ e $\overline{z_1} = \sqrt{2}\,e^{-i\frac{\pi}{4}}$

$$\left(\frac{z_1}{\overline{z_1}}\right)^n = \left(\frac{\sqrt{2}\,e^{i\frac{\pi}{4}}}{\sqrt{2}\,e^{-i\frac{\pi}{4}}}\right)^n = \left(\frac{\sqrt{2}}{\sqrt{2}} e^{i\left(\frac{\pi}{4}+\frac{\pi}{4}\right)}\right)^n = \left(e^{i\frac{\pi}{2}}\right)^n = e^{i\frac{n\pi}{2}}$$

$e^{i\frac{n\pi}{2}}$ é um imaginário puro de coeficiente negativo $\Leftrightarrow$

$$\Leftrightarrow \frac{n\pi}{2} = \frac{3\pi}{2} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow n = 3 + 4k,\ k \in \mathbb{Z}$$

6.3. $\Leftrightarrow z_1^5 + a_1^3 = -b \Leftrightarrow$

$$\Leftrightarrow \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^5 + a\left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^3 = -b \Leftrightarrow$$

$$\Leftrightarrow \left(\sqrt{2}\right)^5 e^{i\frac{5\pi}{4}} + a\left(\sqrt{2}\right)^3 e^{i\frac{3\pi}{4}} = -b \Leftrightarrow$$

$$\Leftrightarrow 4\sqrt{2}\,e^{i\frac{5\pi}{4}} + 2a\sqrt{2}\,e^{i\frac{3\pi}{4}} = -b \Leftrightarrow$$

$$\Leftrightarrow 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) + a\times 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -b \Leftrightarrow$$

$$\Leftrightarrow 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) + a\times 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -b \Leftrightarrow$$

$$\Leftrightarrow -4 - 4i - 2a + 2ai = -b \Leftrightarrow$$

$$\Leftrightarrow (-4-2a) + (2a-4)i = -b + 0i$$

$$\Leftrightarrow \begin{cases} -4-2a = -b \\ 2a - 4 = 0 \end{cases} \Leftrightarrow \begin{cases} -4-4 = -b \\ a = 2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a = 2 \\ b = 8 \end{cases}$$

Portanto, $a = 2$ e $b = 8$

6.4. Os números z_2 e z_3 são raízes de índice 4 do mesmo número complexo se e somente se $(z_2)^4 = (z_3)^4$.

$$(z_2)^4 = \left(\sqrt[4]{2}\,e^{i\frac{2\pi}{3}}\right)^4 = \left(\sqrt[4]{2}\right)^4 e^{i\left(\frac{2\pi}{3}\times 4\right)} = 2e^{i\frac{8\pi}{3}} =$$

$$= 2e^{i\left(2\pi+\frac{2\pi}{3}\right)} = 2e^{i\frac{2\pi}{3}}$$

$$(z_3)^4 = \left(\sqrt[4]{2}\,e^{i\frac{5\pi}{3}}\right)^4 = \left(\sqrt[4]{2}\right)^4 e^{i\left(\frac{5\pi}{3}\times 4\right)} = 2e^{i\frac{20\pi}{3}} = 2e^{i\left(6\pi+\frac{2\pi}{3}\right)} =$$

$$= 2e^{i\frac{2\pi}{3}}$$

Como $(z_2)^4 = (z_3)^4$, efetivamente, os números z_2 e z_3 são s raízes de índice 4 do mesmo número complexo, sendo este número $2e^{i\frac{2\pi}{3}}$.

7. $4z^6 + 256 = 0 \Leftrightarrow z^6 = -\frac{256}{4} \Leftrightarrow z^6 = -64 \Leftrightarrow z = \sqrt[6]{-64}$

$$\Leftrightarrow z = \sqrt[6]{64e^{i\pi}} \Leftrightarrow z = \sqrt[6]{64}\,e^{i\left(\frac{\pi}{6}+\frac{2k\pi}{6}\right)},\ k = 0,1,2,3,4,5 \Leftrightarrow$$

$$\Leftrightarrow x = 2e^{\frac{\pi+2k\pi}{6}},\ k = 0,1,2,3,4,5$$

$$z_0 = 2e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) =$$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

$$z_1 = 2e^{i\frac{\pi}{2}} = 2i$$

$$z_2 = 2e^{i\frac{5\pi}{6}} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) =$$

$$= 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$$z_3 = 2e^{i\frac{7\pi}{6}} = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) =$$

$$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$$

$$z_4 = 2e^{i\frac{3\pi}{2}} = -2i$$

$$z_5 = 2e^{i\frac{11\pi}{6}} = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) =$$

$$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

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8.1. $z^2 = -16 \Leftrightarrow z = \sqrt{-16} \Leftrightarrow z = \sqrt{16e^{i\pi}} \Leftrightarrow$

$$\Leftrightarrow z = \sqrt{16}\,e^{i\left(\frac{\pi}{2}+\frac{2k\pi}{2}\right)},\ k = 0,1 \Leftrightarrow$$

$$\Leftrightarrow z = 4e^{i\frac{\pi}{2}} \vee z = e^{i\frac{3\pi}{2}}$$

8.2. $z^2 = 4 \Leftrightarrow z = \sqrt{4} \Leftrightarrow z = \sqrt{4e^{i0}} \Leftrightarrow$

$$\Leftrightarrow z = 2 \vee z = -2 \Leftrightarrow$$

$$\Leftrightarrow z = 2e^{i0} \vee z = 2e^{i\pi}$$

8.3. $z^4 = -1 \Leftrightarrow z = \sqrt[4]{-1} \Leftrightarrow z = \sqrt[4]{e^{i\pi}} \Leftrightarrow$

$$\Leftrightarrow z = \sqrt[4]{1}\,e^{i\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)},\ k = 0,1,2,3 \Leftrightarrow$$

$$\Leftrightarrow z = e^{i\frac{\pi}{4}} \vee z = e^{i\frac{3\pi}{4}} \vee z = e^{i\frac{5\pi}{4}} \vee z = e^{i\frac{7\pi}{4}}$$

8.4. $z^5 = 32e^{i\frac{\pi}{5}} \Leftrightarrow z = \sqrt[5]{32e^{i\frac{\pi}{5}}} \Leftrightarrow$

$$\Leftrightarrow z = \sqrt[5]{32}\,e^{i\left(\frac{\pi}{25}+\frac{2k\pi}{5}\right)},\ k = 0,1,2,3,4 \Leftrightarrow$$

$$\Leftrightarrow z = 2e^{i\frac{\pi}{25}} \vee z = 2e^{i\frac{11\pi}{25}} \vee z = 2e^{i\frac{21\pi}{5}} \vee z = 2e^{i\frac{31\pi}{25}} \vee z = 2e^{i\frac{41\pi}{25}}$$

8.5. Seja $w = -3\sqrt{3} + 9\mathrm{i}$

$$|w| = \sqrt{\left(-3\sqrt{3}\right)^2 + 9^2} = \sqrt{27+81} = \sqrt{108} = 6\sqrt{3}$$

Sendo $\operatorname{Arg}(w) = \theta$, tem-se $\tan\theta = \dfrac{9}{-3\sqrt{3}} \wedge \theta \in 2.^{\text{o}}\text{Q}$, ou seja, $\tan\theta = -\sqrt{3} \wedge \theta \in 2.^{\text{o}}\text{Q}$, então $\theta = \dfrac{2\pi}{3}$.

Portanto, $w = 6\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$.

Assim:

$$z^3 = -3\sqrt{3} + 9\mathrm{i} \Leftrightarrow z = \sqrt[3]{-3\sqrt{3}+9\mathrm{i}} \Leftrightarrow z = \sqrt[3]{6\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}} \Leftrightarrow$$

$$\Leftrightarrow z = \sqrt[3]{6\sqrt{3}}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)},\ k = 0,1,2 \Leftrightarrow$$

$$\Leftrightarrow z = \sqrt[3]{\sqrt{6^2\times 3}}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{9}+\frac{6k\pi}{9}\right)},\ k = 0,1,2 \Leftrightarrow$$

$$\Leftrightarrow z = \sqrt[6]{108}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi+6k\pi}{9}\right)},\ k = 0,1,2 \Leftrightarrow$$

$$\Leftrightarrow z = \sqrt[6]{108}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{9}} \vee z = \sqrt[6]{108}\,\mathrm{e}^{\mathrm{i}\frac{8\pi}{9}} \vee z = \sqrt[6]{108}\,\mathrm{e}^{\mathrm{i}\frac{14\pi}{9}}$$

8.6. Seja $w = \dfrac{128}{-1+\mathrm{i}} = \dfrac{128\mathrm{e}^{\mathrm{i}0}}{\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}} = \dfrac{128}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} = 64\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}$

Assim, vem:

$$z^6 = \frac{128}{-1+\mathrm{i}} \Leftrightarrow z^6 = 64\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} \Leftrightarrow z = \sqrt[6]{64\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}} \Leftrightarrow$$

$$\Leftrightarrow z = \sqrt[6]{64\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{8}+\frac{k\pi}{3}\right)},\ k = 0,1,2,3,4,5 \Leftrightarrow$$

$$\Leftrightarrow z = 2\sqrt[6]{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{8}+\frac{k\pi}{3}\right)},\ k = 0,1,2,3,4,5 \Leftrightarrow$$

$$\Leftrightarrow z = 2\sqrt[12]{2}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{8}+\frac{k\pi}{3}\right)},\ k = 0,1,2,3,4,5 \Leftrightarrow$$

$$\Leftrightarrow z = \sqrt[12]{2}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}} \vee z = 2\sqrt[12]{2}\,\mathrm{e}^{\mathrm{i}\frac{5\pi}{24}} \vee z = 2\sqrt[12]{2}\,\mathrm{e}^{\mathrm{i}\frac{13\pi}{24}} \vee$$

$$\vee z = 2\sqrt[12]{2}\,\mathrm{e}^{\mathrm{i}\frac{7\pi}{8}} \vee z = 2\sqrt[12]{2}\,\mathrm{e}^{\mathrm{i}\frac{29\pi}{24}} \vee z = 2\sqrt[12]{2}\,\mathrm{e}^{\mathrm{i}\frac{37\pi}{24}}$$

9.1. Tem-se que $-8\mathrm{i} = 8\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$, pelo que:

$$\sqrt[3]{-8\mathrm{i}} = \sqrt[3]{8\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}} =$$

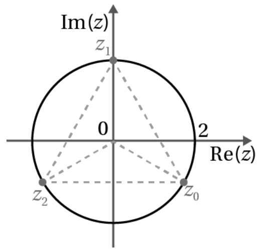
$$= \sqrt[3]{8}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}+\frac{2k\pi}{3}\right)},\ k = 0,1,2 =$$

$$= 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}+\frac{2k\pi}{3}\right)},\ k = 0,1,2$$

As raízes cúbicas de $-8\mathrm{i}$ são:

$$z_0 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\ ;\ z_1 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\ ;\ z_2 = 2\mathrm{e}^{\mathrm{i}\frac{7\pi}{6}} .$$

A sua representação no plano complexo corresponde aos três vértices de um triângulo equilátero inscrito numa circunferência de raio 2 e centro na origem do referencial.



9.2. $-256 = 256\mathrm{e}^{\mathrm{i}\pi}$, pelo que:

$$\sqrt[4]{-256} = \sqrt[4]{256\mathrm{e}^{\mathrm{i}\pi}} =$$

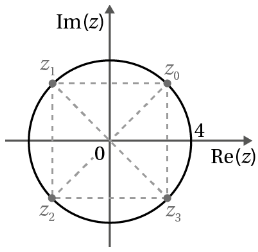
$$= \sqrt[4]{256}\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)},\ k = 0,1,2,3 =$$

$$= 4\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}+\frac{12\pi}{2}\right)},\ k = 0,1,2,3$$

As raízes quartas de -256 são:

$$z_0 = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\ ;\ z_1 = 4\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\ ;\ z_2 = 4\mathrm{e}^{\mathrm{i}\frac{5\pi}{4}}\ ;\ z_3 = 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{4}} .$$

A sua representação no plano complexo aos quatro vértices de um quadrado inscrito numa circunferência de raio 4 e centro na origem do referencial.



9.3. Seja $w = -16\sqrt{2} + 16\sqrt{2}$

$$|w| = \sqrt{\left(-16\sqrt{2}\right)^2 + \left(16\sqrt{2}\right)^2} = \sqrt{1024} = 32$$

Sendo $\operatorname{Arg} w = \theta$, tem-se $\tan\theta = \dfrac{16\sqrt{2}}{-16\sqrt{2}} \wedge \theta \in 2.^{\text{o}}\text{Q}$, isto é,

$\tan\theta = -1 \wedge \theta \in 2.^{\text{o}}\text{Q}$, então $\theta = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$.

Portanto, $w = 32\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$.

Assim:

$$\sqrt[5]{-16\sqrt{2}+16\sqrt{2}\,\mathrm{i}} = \sqrt[5]{32\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}} = \sqrt[5]{32}\,\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{20}+\frac{2k\pi}{5}\right)},\ k = 0,1,2,3,4$$

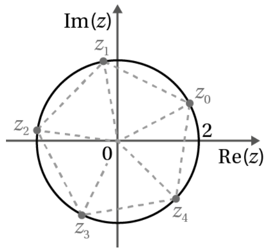
$$= 2\mathrm{e}^{\mathrm{i}\left(\frac{3\pi+8k\pi}{20}\right)},\ k = 0,1,2,3,4$$

As raízes quintas de $-16\sqrt{2}+16\sqrt{2}\,\mathrm{i}$ são:

$$z_0 = 2\mathrm{e}^{\mathrm{i}\frac{3\pi}{20}}\ ;\ z_1 = 2\mathrm{e}^{\mathrm{i}\frac{11\pi}{20}}\ ;\ z_2 = 2\mathrm{e}^{\mathrm{i}\frac{19\pi}{20}}\ ;\ z_3 = 2\mathrm{e}^{\mathrm{i}\frac{27\pi}{20}}\ ;$$

$$z_4 = 2\mathrm{e}^{\mathrm{i}\frac{7\pi}{4}}$$

A sua representação no plano complexo corresponde aos cinco vértices de um pentágono inscrito numa circunferência de raio 2 e centro na origem do referencial.



9.4. $\frac{1}{\mathrm{i}} = \frac{\mathrm{i}}{\mathrm{i}^2} = \frac{\mathrm{i}}{-1} = -\mathrm{i} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$

Assim:

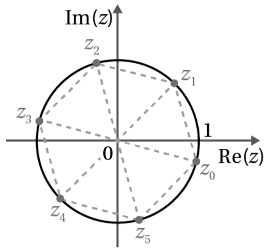
$\sqrt[6]{\frac{1}{\mathrm{i}}} = \sqrt[6]{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}} = \sqrt[6]{1}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{12}+\frac{2k\pi}{6}\right)}, k = 0, 1, 2, 3, 4, 5$

$= \mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{12}+\frac{k\pi}{3}\right)}, k = 0, 1, 2, 3, 4, 5$

As raízes sextas de $\frac{1}{\mathrm{i}}$ são: $z_0 = \mathrm{e}^{-\mathrm{i}\frac{\pi}{12}}$; $z_1 = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$; $z_2 = \mathrm{e}^{\mathrm{i}\frac{7\pi}{12}}$;

$z_3 = \mathrm{e}^{\mathrm{i}\frac{11\pi}{12}}$; $z_4 = \mathrm{e}^{\mathrm{i}\frac{5\pi}{4}}$; $z_5 = \mathrm{e}^{\mathrm{i}\frac{19\pi}{12}}$

A sua representação no plano complexo corresponde aos seis vértices de um hexágono regular inscrito numa circunferência de raio 1 e centro na origem do referencial.



10.1. Os módulos de qualquer raiz de vértice n de w são iguais, neste caso, $|z_1| = 2$, e os argumentos das n raízes correspondentes a valores de k consecutivos são termos de uma progressão aritmética de razão $\frac{2\pi}{n}$, neste caso $\frac{2\pi}{4} = \frac{\pi}{2}$.

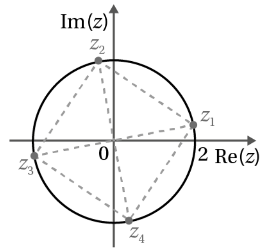
Logo, se $z_1 = \mathrm{e}^{\mathrm{i}\frac{\pi}{12}}$ é uma das raízes quartas de um certo número complexo w, então as restantes três raízes são:

$z_2 = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{12}+\frac{\pi}{2}\right)} = 2\mathrm{e}^{\mathrm{i}\frac{7\pi}{12}}$

$z_3 = 2\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{12}+\frac{\pi}{2}\right)} = 2\mathrm{e}^{\mathrm{i}\frac{13\pi}{12}}$

$z_4 = 2\mathrm{e}^{\mathrm{i}\left(\frac{13\pi}{12}+\frac{\pi}{2}\right)} = 2\mathrm{e}^{\mathrm{i}\frac{19\pi}{12}}$

10.2. Representemos geometricamente as raízes quartas de w.



Os afixos das raízes quartas de w são, portanto, os vértives de um quadrado inscrito numa circunferência de raio 2 e centro na origem do referencial.

A diagonal do quadrado é igual a 4, pois é igual a $|z_1| + |z_3|$.

Se a for a medida do lado do quadrado, vem

$a^2 + a^2 = 4^2 \Leftrightarrow 2a^2 = 16 \Leftrightarrow a^2 = 8$.

Logo, a área do quadrado é $a^2 = 8$.

10.3. Como as raízes são simétricas duas a duas, isto é, $z_1 = -z_3$ e $z_2 = -z_4$, tem-se que $z_1 + z_3 = 0$ e $z_2 + z_4 = 0$ e, portanto,

$z_1 + z_2 + z_3 + z_4 = 0$.

11.1. $z^4 - \mathrm{i}\,z + \sqrt{3}z = 0 \Leftrightarrow z\left(z^3 - \mathrm{i} + \sqrt{3}\right) = 0 \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z^3 = -\sqrt{3} + \mathrm{i} \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = \sqrt[3]{-\sqrt{3} + \mathrm{i}}$

Vamos escrever $-\sqrt{3} + \mathrm{i}$ na forma trigonométrica:

$\left|-\sqrt{3} + \mathrm{i}\right| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$

Sendo $\operatorname{Arg}\left(-\sqrt{3} + \mathrm{i}\right) = \theta$, tem-se $\tan\theta = \frac{1}{-\sqrt{3}} \wedge \theta \in 2.^{\text{o}}\text{Q}$,

isto é, $\tan\theta = -\frac{\sqrt{3}}{3} \wedge \theta \in 2.^{\text{o}}\text{Q}$, então, $\theta = \frac{5\pi}{6}$. Portanto,

$-\sqrt{3} + \mathrm{i} = 2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$.

$z = 0 \vee z = \sqrt[3]{-\sqrt{3} + \mathrm{i}} \Leftrightarrow z = 0 \vee z = \sqrt[3]{2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}} \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = \sqrt[3]{2}\,\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{18}+\frac{2k\pi}{3}\right)}, k = 0, 1, 2 \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = \sqrt[3]{2}\,\mathrm{e}^{\mathrm{i}\frac{5\pi}{18}} \vee z = \sqrt[3]{2}\,\mathrm{e}^{\mathrm{i}\frac{17\pi}{18}} \vee z = \sqrt[3]{2}\,\mathrm{e}^{\mathrm{i}\frac{29\pi}{18}}$

11.2. $z^5 = 32\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} = 0 \Leftrightarrow z^5 = -32\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} \Leftrightarrow z^5 = 32\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}+\pi\right)} \Leftrightarrow$

$\Leftrightarrow z^5 = 32\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}} \Leftrightarrow z = \sqrt[5]{32\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}} \Leftrightarrow$

$\Leftrightarrow z = \sqrt[5]{32}\,\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{15}+\frac{2k\pi}{5}\right)}, k = 0, 1, 2, 3, 4 \Leftrightarrow$

$\Leftrightarrow 2\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{15}+\frac{2k\pi}{3}\right)}, k = 0, 1, 2, 3, 4 \Leftrightarrow$

$\Leftrightarrow z = 2\mathrm{e}^{\mathrm{i}\frac{4\pi}{5}} \vee z = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} \vee z = 2\mathrm{e}^{\mathrm{i}\frac{16\pi}{15}} \vee z = 2\mathrm{e}^{\mathrm{i}\frac{22\pi}{15}} \vee z = 2\mathrm{e}^{\mathrm{i}\frac{32\pi}{15}}$

11.3. Seja $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, onde $r \in \mathbb{R}^+ \wedge \theta \in \mathbb{R}$, então, $\overline{z} = r\,\mathrm{e}^{-\mathrm{i}\theta}$.

$z^3 = 3\mathrm{i}\,\overline{z} = 0 \Leftrightarrow$

$\Leftrightarrow z^3 = -3\mathrm{i}\,\overline{z} \Leftrightarrow$

$\Leftrightarrow \left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)^3 = -3^{-\mathrm{i}\frac{\pi}{2}} \times r\,\mathrm{e}^{-\mathrm{i}\theta} \Leftrightarrow$

$\Leftrightarrow r^3\,\mathrm{e}^{3\mathrm{i}\theta} = 3r\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}-\theta\right)} \Leftrightarrow$

$\Leftrightarrow \begin{cases} r = 3r = 0 \\ 4\theta = -\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z} \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} r\left(r^2-3\right)=0 \\ \theta=-\frac{\pi}{8}+\frac{2k\pi}{4},\ k=0,1,2,3 \end{cases}$$

Para $k=4$ obtém-se a mesma solução que para $k=0$.

$$\Leftrightarrow \begin{cases} r=0 \vee r=\sqrt{3} \\ \theta=-\frac{\pi}{8}+\frac{k\pi}{2},\ k=0,1,2,3 \end{cases}$$

Para $r=0$ obtém-se apenas a solução $z=0$.

$$\Leftrightarrow z=0 \vee z=\sqrt{3}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}} \vee z=\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{3\pi}{8}} \vee z=\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{7\pi}{8}} \vee z=\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{11\pi}{8}}$$

11.4. $\overline{z}^3-9z=0 \Leftrightarrow \overline{z}^3=9z$, sendo $z=r\,\mathrm{e}^{\mathrm{i}\theta}$, com $r \in \mathbb{R}_0^+ \wedge \theta \in \mathbb{R}$ e substituindo na equação, tem-se:

$$\overline{z}^3=9z \Leftrightarrow \left(r\,\mathrm{e}^{-\mathrm{i}\theta}\right)^3=9r\,\mathrm{e}^{\mathrm{i}\theta} \Leftrightarrow r^3\,\mathrm{e}^{-\mathrm{i}3\theta}=9r\,\mathrm{e}^{\mathrm{i}\theta} \Leftrightarrow$$

$$\Leftrightarrow r^3=9r \wedge -3\theta=\theta+2k\pi,\ k\in\mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow r^3-9r=0 \wedge -4\theta=2k\pi,\ k\in\mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow r\left(r^2-9\right)=0 \wedge \theta=-\frac{k\pi}{2},\ k\in\mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow \left(r=0 \vee r=3\right) \wedge \theta=-\frac{k\pi}{2},\ k\in\mathbb{Z} \Leftrightarrow \text{, pois } r\in\mathbb{R}_0^+$$

$$\Leftrightarrow \left(r=3 \wedge \theta=-\frac{k\pi}{2},\ k=0,1,2,3\right)$$

As soluções da equação são:

$$z=0 \vee z=3\mathrm{e}^{\mathrm{i}0} \vee z=3\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} \vee z=3\mathrm{e}^{-\mathrm{i}\pi} \vee z=3\mathrm{e}^{-\mathrm{i}\frac{3\pi}{2}}$$

11.5. As soluções da equação são as raízes cúbicas de $(2+\mathrm{i})^3$, uma delas é $2+\mathrm{i}$; as outras obtêm-se multiplicando $2+\mathrm{i}$ por $\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ e por $\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right)}=\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}$.

Então, $\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$ e $\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$. Assim, as soluções da equação são:

- $z=2+\mathrm{i}$
- $z=(2+\mathrm{i})\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)=-1+\sqrt{3}\mathrm{i}-\frac{1}{2}\mathrm{i}+\frac{\sqrt{3}}{2}\mathrm{i}^2=$

  $=\left(-1-\frac{\sqrt{3}}{2}\right)+\left(\sqrt{3}-\frac{1}{2}\right)\mathrm{i}$
- $z=(2+\mathrm{i})\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)=-1-\sqrt{3}\mathrm{i}-\frac{1}{2}\mathrm{i}-\frac{\sqrt{3}}{2}\mathrm{i}^2=$

  $=\left(-1+\frac{\sqrt{3}}{2}\right)+\left(-\sqrt{3}-\frac{1}{2}\right)\mathrm{i}$

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12.1. Os arcos da circunferência compreendidos entre dois vértices consecutivos, do quadrado, são iguais. Cada um deles tem, amplitude igual a $\frac{2\pi}{4}=\frac{\pi}{2}$.

Por outro lado, a imagem geométrica de $2\mathrm{e}^{\mathrm{i}\frac{\pi}{5}}$ é o ponto A (único vértice do quadrado que se encontra no primeiro quadrante e $0<\frac{\pi}{5}<\frac{\pi}{2}$).

Assim, os números complexos cujas imagens geométricas são os pontos B, C e D, são respetivamente,

$$z_B=2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{5}+\frac{\pi}{2}\right)}=2\mathrm{e}^{\mathrm{i}\frac{7\pi}{10}}$$

$$z_C=2\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{10}+\frac{\pi}{2}\right)}=2\mathrm{e}^{\mathrm{i}\frac{6\pi}{5}}$$

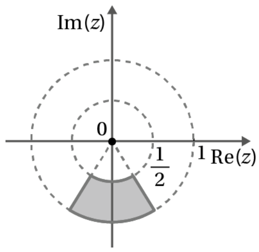
$$z_D=2\mathrm{e}^{\mathrm{i}\left(\frac{6\pi}{5}+\frac{\pi}{2}\right)}=2\mathrm{e}^{\mathrm{i}\frac{17\pi}{10}}$$

12.2. $|z|<2 \wedge \frac{7\pi}{10}<\operatorname{Arg}(z)<\frac{6\pi}{5}$

13. A condição $\frac{1}{2}\leq|z|\leq 1$ corresponde à coroa circular delimitada pelas circunferências de centro na imagem e raios iguais a $\frac{1}{2}$ e 1.

A condição $-\frac{5\pi}{8}\leq\operatorname{Arg}(z)\leq-\frac{3\pi}{8}$ corresponde à região do plano complexo compreendidas entre as semirretas definidas por $\operatorname{Arg}(z)=-\frac{5\pi}{8}$ e $\operatorname{Arg}(z)=-\frac{3\pi}{8}$.

Assim, a região do plano definida pela condição, em $\mathbb{C}$, $\frac{1}{2}\leq|z|\leq 1 \wedge -\frac{5\pi}{8}<\operatorname{Arg}(z)\leq-\frac{3\pi}{8}$ é:



$$-\frac{3\pi}{8}-\left(-\frac{5\pi}{8}\right)=\frac{2\pi}{8}$$

A área desta região é igual à oitava parte da área da coroa circular que, por sua vez, é igual à diferença entre as áreas dos dois círculos (o maior, de raio 1, e o menor, de raio $\frac{1}{2}$).

Portanto, a área da região é igual a

$$\frac{1}{8}\times\left[\pi\times1^2-\pi\times\left(\frac{1}{2}\right)^2\right]=\frac{1}{8}\times\frac{3\pi}{4}=\frac{3\pi}{32}\ \text{u.a.}$$

14.1. $z^3=z_1 \Leftrightarrow z^3=-\frac{27}{8}\mathrm{i} \Leftrightarrow z^3=\frac{27}{8}\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}} \Leftrightarrow$

$$\Leftrightarrow z=\sqrt[3]{\frac{27}{8}\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}}$$

$$\Leftrightarrow z=\sqrt[3]{\frac{27}{8}}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{6}+\frac{2k\pi}{3}\right)},\ k=0,1,2 \Leftrightarrow$$

$$\Leftrightarrow z=\sqrt[3]{\left(\frac{3}{2}\right)^3}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\frac{2k\pi}{3}\right)},\ k=0,1,2 \Leftrightarrow$$

$\Leftrightarrow z=\frac{3}{2}e^{i\frac{\pi}{2}} \vee z=\frac{3}{2}e^{i\frac{7\pi}{6}} \vee z=\frac{3}{2}e^{i\frac{11\pi}{6}} \Leftrightarrow$

$\Leftrightarrow z=\frac{3}{2}i \vee z=\frac{3}{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)\vee$

$\vee z=\frac{3}{2}\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)\Leftrightarrow$

$\Leftrightarrow z=\frac{3}{2}i \vee z=\frac{3}{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\vee z=\frac{3}{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\Leftrightarrow$

$\Leftrightarrow z=\frac{3}{2}i \vee z=-\frac{3\sqrt{3}}{4}-\frac{3}{4}i \vee z=\frac{3\sqrt{3}}{4}-\frac{3}{4}i \Leftrightarrow$

14.2. ▪ $\mathrm{Im}(z-\bar{z})\leq z\bar{z}$, substituindo, z por $x+yi$ e $\bar{z}$ por $x-yi$, vem:

$\mathrm{Im}\left[(x+yi)-(x-yi)\right]\leq(x+yi)(x-yi)\Leftrightarrow$

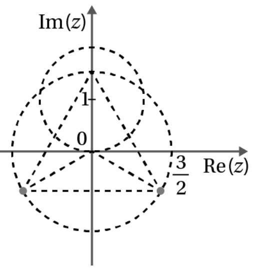
$\Leftrightarrow \mathrm{Im}(2yi)\leq x^2+y^2 \Leftrightarrow$

$\Leftrightarrow 2y\leq x^2+y^2 \Leftrightarrow$

$\Leftrightarrow x^2+y^2-2y+1\geq 1 \Leftrightarrow$

$\Leftrightarrow x^2+(y-1)^2\geq 1$, define que o complementar do círculo aberto de centro no ponto de coordenadas $(0,1)$ e raio igual a 1.

▪ $8z^3+27i=0 \Leftrightarrow z^3=-\frac{27}{8}i$, por **14.1.**, sabemos que define os três vértices de um, triângulo equilátero, inscrito numa circunferência de raio $\frac{3}{2}$ e centro na origem.



O conjunto definido pela condição:

$$\mathrm{Im}(z-\bar{z})\leq z\bar{z} \wedge 8z^3+27i=0$$

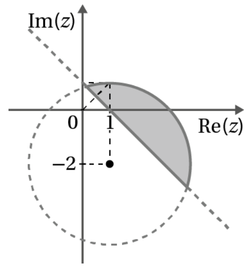
tem apenas dois elementos: os dois pontos assinalados a cor vermelha na figura.

15. $|z-z_2|\leq 3 \Leftrightarrow |z-(1-2i)|\leq 3$, define o círculo de centro no ponto de coordenadas $(1,-2)$ e raio 3.

$|z|\geq|z+z_1| \Leftrightarrow |z|\geq|z+(-1-i)|$

$\Leftrightarrow |z-(0+0i)|\geq|z-(1+i)|$

define o semiplano fechado (inclui fronteira) delimitado pela mediatriz do segmento de reta $[OA]$, sendo $O(0,0)$ e $A(1,1)$, e que contém o ponto A.



Ficha de teste 9

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1. Seja $z=re^{i\theta}$, com $r\in\mathbb{R}^+ \wedge \theta\in\left]\frac{\pi}{2},\pi\right[$.

Então, $z^3=\left(re^{i\theta}\right)^3=r^3e^{i3\theta}$.

$\frac{\pi}{2}<\theta<\pi \Leftrightarrow \frac{3\pi}{2}<3\theta<3\pi$

Logo, se $3\theta\in\left]\frac{3\pi}{2},3\pi\right[$, z^3 não pertence ao 3.º quadrante.

Resposta: **(C)**

2. $\frac{z^3}{e^{36n+5}}=\frac{\left(e^{i\theta}\right)^3}{i^{(4\times9)n}\times i^4\times i}=\frac{\left(e^{i\theta}\right)^3}{i}=\frac{e^{i3\theta}}{e^{i\frac{\pi}{2}}}=e^{i\left(3\theta-\frac{\pi}{2}\right)}$

$\frac{z^3}{i^{36n+5}}$ é um número real, se e só se, $3\theta-\frac{\pi}{2}=k\pi, k\in\mathbb{Z}$, portanto,

$3\theta-\frac{\pi}{2}=,k\pi, k\in\mathbb{Z}\Leftrightarrow 3\theta=\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\Leftrightarrow\theta=\frac{\pi}{6}+\frac{k\pi}{3}, k\in\mathbb{Z}$

Fazendo $k=0$, vem $\theta=\frac{\pi}{6}$. Como $\frac{\pi}{6}\in\left]0,\frac{\pi}{2}\right[$ está encontrado o valor pedido.

Resposta: **(A)**

3. Temos que $|z+w|\leq|z|+|w|$, portanto:

$|z+w|\leq 5+15 \Leftrightarrow |z+w|<20$

Resposta: **(D)**

4. O número complexo z_1 é um número real quando e apenas quando

$\mathrm{Im}(z)=0 \Leftrightarrow 4-a^2=0 \Leftrightarrow a=-2 \vee a=2$

Resposta: **(B)**

5. O eixo imaginário pode se definido pela condição

$\mathrm{Arg}(z)=\frac{\pi}{2}\vee\mathrm{Arg}(z)=-\frac{\pi}{2}\Leftrightarrow|\mathrm{Arg}(z)|=\frac{\pi}{2}$

Resposta: **(C)**

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6.1. O inverso de z_2 é $\frac{1}{z_2}$.

Determinemos, inicialmente, z_2, na forma trigonométrica.

$$z_2=\frac{(z_1)^2+4i-3\sqrt{3}e^{i\frac{\pi}{2}}}{2e^{i\frac{\pi}{2}}}=\frac{(2-i)^2+4i+3\sqrt{3}i}{3e^{i\frac{\pi}{12}}}=$$

$$=\frac{4+4i+i^2+4i+3\sqrt{3}i}{3e^{i\frac{\pi}{12}}}=\frac{3+3\sqrt{3}i}{3e^{i\frac{\pi}{12}}}=$$

$$= \frac{3\left(1+\sqrt{3}\,\mathrm{i}\right)}{3\,\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}} = \frac{1+\sqrt{3}\,\mathrm{i}}{\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}} =$$

$$= \frac{2\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}}{\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}} = 2\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}-\frac{\pi}{12}\right)} = 2\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$$

$|1+\sqrt{3}\mathrm{i}| = \sqrt{1+3} = 2$

$\operatorname{Arg}\left(1-\sqrt{3}\mathrm{i}\right) = \theta$

$\tan\theta = \sqrt{3} \wedge \theta \in 1.^{\text{o}}\,\mathrm{Q}$

$\theta = \frac{\pi}{3}$

Assim:

$$\frac{1}{z_2} = \frac{1}{2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}} = \frac{\mathrm{e}^{\mathrm{i}0}}{2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}} = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} = \frac{1}{2}\left(\cos\left(-\frac{\pi}{4}\right)+\sin\left(-\frac{\pi}{4}\right)\right) =$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}\right) = \frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{4}\mathrm{i}$$

6.2. $\left(z_1\right)^2 z - 2z_1 + z = 0 \Leftrightarrow (2-\mathrm{i})^2 z - 2(2-\mathrm{i}) + z = 0 \Leftrightarrow$

$$\Leftrightarrow (2-\mathrm{i})^2 z + z = 2(2-\mathrm{i}) \Leftrightarrow \left[(2-\mathrm{i})^2+1\right]z = 4-2\mathrm{i} \Leftrightarrow$$

$$\Leftrightarrow \left(4-4\mathrm{i}+\mathrm{i}^2+1\right)z = 4-2\mathrm{i} \Leftrightarrow (4-4\mathrm{i})z = 4-2\mathrm{i} \Leftrightarrow$$

$$\Leftrightarrow z = \frac{4-2\mathrm{i}}{4-4\mathrm{i}} \Leftrightarrow z = \frac{2(2-\mathrm{i})}{2(2-2\mathrm{i})} \Leftrightarrow z = \frac{2-\mathrm{i}}{2-2\mathrm{i}} \Leftrightarrow$$

$$\Leftrightarrow z = \frac{(2-\mathrm{i})(2+2\mathrm{i})}{(2-2\mathrm{i})(2+2\mathrm{i})} \Leftrightarrow$$

$$\Leftrightarrow z = \frac{4+4\mathrm{i}-2\mathrm{i}-2\mathrm{i}^2}{4-4\mathrm{i}^2} \Leftrightarrow z = \frac{4+2\mathrm{i}+2}{8} \Leftrightarrow z = \frac{6+2\mathrm{i}}{8} \Leftrightarrow$$

$$\Leftrightarrow z = \frac{3}{4}+\frac{1}{4}\mathrm{i}$$

6.3. Pretende-se determinar $\sqrt[4]{2z_2}$.

Como $z_2 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$, vem:

$$\sqrt[4]{2z_2} = \sqrt[4]{2\times 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}} = \sqrt[4]{4\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}} = \sqrt[4]{4}\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{16}+\frac{2k\pi}{4}\right)},\ k = 0,1,2,3$$

$$= \sqrt[4]{2^2}\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{16}+\frac{k\pi}{2}\right)},\ k = 0,1,2,3$$

$$= \sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{16}+\frac{k\pi}{2}\right)},\ k = 0,1,2,3$$

As raízes de índice 4 do número complexo $w = 2z_2$ são:

$$\sqrt{2}\,\mathrm{e}^{\frac{\pi}{16}},\ \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{9\pi}{16}},\ \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{17\pi}{16}} \text{ e } \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{25}{16}\pi}$$

7.1. $3z^2+2z+7 = 0 \Leftrightarrow z = \frac{-2\pm\sqrt{4-4\times 3\times 7}}{2\times 3} \Leftrightarrow$

$$\Leftrightarrow z = \frac{-2\pm\sqrt{-80}}{6} \Leftrightarrow z = \frac{-2\pm\sqrt{80}\,\mathrm{i}}{6} \Leftrightarrow$$

$$\Leftrightarrow z = \frac{-2\pm 4\sqrt{5}\,\mathrm{i}}{6} \Leftrightarrow z = -\frac{1}{3}+\frac{2\sqrt{5}}{3}\mathrm{i} \vee z = -\frac{1}{3}-\frac{2\sqrt{5}}{3}\mathrm{i}$$

As soluções da equação são:

$$z = -\frac{1}{3}+\frac{2\sqrt{5}}{3}\mathrm{i} \vee z = -\frac{1}{3}-\frac{2\sqrt{5}}{3}\mathrm{i}\,.$$

7.2. $z^2 = \left(-1-\mathrm{i}\sqrt{3}\right)\bar{z} \Leftrightarrow z^2 = \left(2\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}\right)\bar{z}$, substituindo z por $r\mathrm{e}^{\mathrm{i}\theta}$, onde $r \in \mathbb{R}_0^+$ e $\theta \in \mathbb{R}$, então, $\bar{z} = r\mathrm{e}^{-\mathrm{i}\theta}$, vem:

$$\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^2 = \left(2\mathrm{e}^{\mathrm{i}\frac{4\pi}{3}}\right)\left(r\mathrm{e}^{-\mathrm{i}\theta}\right) \Leftrightarrow$$

$$\Leftrightarrow r^2\mathrm{e}^{\mathrm{i}2\theta} = 2r\,\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}-\theta\right)}$$

$$\Leftrightarrow \begin{cases} r^2 = 2r \\ 2\theta = \frac{4\pi}{3}-\theta+2k\pi,\ k\in\mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r^2-2r = 0 \\ 3\theta = \frac{4\pi}{3}+2k\pi,\ k\in\mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r(r-2) = 0 \\ \theta = \frac{4\pi}{9}+\frac{2k\pi}{3},\ k = 0,1,2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r = 0 \vee r = 2 \\ \theta = \frac{4\pi}{9}+\frac{2k\pi}{3},\ k = 0,1,2 \end{cases}$$

$$\Leftrightarrow z = 0 \vee z = 2\mathrm{e}^{\mathrm{i}\frac{4\pi}{9}} \vee z = 2\mathrm{e}^{\mathrm{i}\frac{10\pi}{9}} \vee z = 2\mathrm{e}^{\mathrm{i}\frac{16\pi}{9}}$$

8. Tem-se que:

$$\sqrt[n]{1} = \sqrt[n]{1\mathrm{e}^{\mathrm{i}0}} = \sqrt[n]{1}\,\mathrm{e}^{\mathrm{i}\left(\frac{2k\pi}{n}\right)} = \mathrm{e}^{\mathrm{i}\frac{2k\pi}{n}},\ k = 0,1,2,\ldots,n-1$$

Como $\mathrm{e}^{\mathrm{i}\frac{2k\pi}{n}} = \left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^k$, pretende-se determinar a soma de n termos consecutivos de uma progressão geométrica de primeiro termo $\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^0 = 1$ e razão $\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}$.

Recorrendo à fórmula da soma de n termos consecutivos de uma progressão geométrica, tem-se:

$$\sum_{k=0}^{n-1}\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^k = 1\times\frac{1-\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^n}{1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}} = \frac{1-\mathrm{e}^{\mathrm{i}2\pi}}{1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}} = \frac{1-1}{1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}} = 0$$

9.1. Seja $w = -3+3\mathrm{i}$

$$|w| = \sqrt{(-3)^2+3^2} = \sqrt{9+9} =$$

$$= \sqrt{18} = 3\sqrt{2}$$

Sendo $\operatorname{Arg}(w) = \theta$, tem-se $\tan\theta = \frac{3}{-3} \wedge \theta \in 2.^{\text{o}}\,\mathrm{Q}$, isto é, $\tan\theta = -1 \wedge \theta \in 2.^{\text{o}}\,\mathrm{Q}$, então $\theta = \pi-\frac{\pi}{4} = \frac{3\pi}{4}$.

Portanto, $w = 3\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$.

Assim:

$$z_1 = (-3+3\mathrm{i})^4 \Leftrightarrow$$

$$\Leftrightarrow z_1 = \left(3\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\right)^4 \Leftrightarrow$$

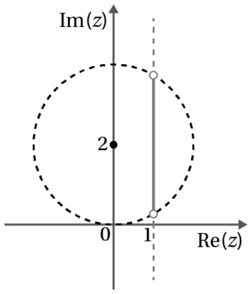
$$\Leftrightarrow z_1 = \left(3\sqrt{2}\right)^4\mathrm{e}^{\mathrm{i}3\pi} \Leftrightarrow$$

$$\Leftrightarrow z_1 = 324\mathrm{e}^{\mathrm{i}\pi}$$

O módulo de z_1 é 324 o argumento principal é π .

9.2.

- $\left|z-2\text{i}\right|<2 \Leftrightarrow \left|z-\left(0+2\text{i}\right)\right|<2$, define o interior de um círculo de centro no ponto de coordenadas $\left(0\,,2\right)$ e raio 2.
- $\text{Re}\left(z\right)=1$,define a reta vertical que passa pelo ponto de coordenadas $\left(1\,,0\right)$.



Temos que $z_1=324\text{e}^{\text{i}\pi}=-324$, pelo que, o seu afixo se situa no eixo real negativo, logo não pertence à condição $\left|z-2\text{i}\right|<2 \wedge \text{Re}\left(z\right)=1$.

9.3. $\left(\left|z_1\right|\right)^{-1}\left(648-\sqrt{3}\,\text{i}\times 648\right)=z\,\text{i}+z \Leftrightarrow$

$$\Leftrightarrow \left(324\right)^{-1}\left(648-\sqrt{3}\,\text{i}\times 648\right)=z\,\text{i}+z \Leftrightarrow$$

$$\Leftrightarrow \frac{648}{324}\left(1-\sqrt{3}\,\text{i}\right)=z\,\text{i}+z \Leftrightarrow$$

$$\Leftrightarrow 2-2\sqrt{3}\,\text{i}=z\left(\text{i}+1\right) \Leftrightarrow$$

$$\Leftrightarrow z=\frac{2-2\sqrt{3}\text{i}}{1+\text{i}} \Leftrightarrow$$

$$\Leftrightarrow z=\frac{\left(2-2\sqrt{3}\text{i}\right)\left(1-\text{i}\right)}{\left(1+\text{i}\right)\left(1-\text{i}\right)} \Leftrightarrow$$

$$\Leftrightarrow z=\frac{2-2\text{i}-2\sqrt{3}\text{i}-2\sqrt{3}}{1+\text{i}}=$$

$$=\frac{\left(2-2\sqrt{3}\right)-\left(2+2\sqrt{3}\right)\text{i}}{2}=$$

$$=1-\sqrt{3}-\left(1+\sqrt{3}\right)\text{i}$$

A solução da equação é $z=1-\sqrt{3}-\left(1+\sqrt{3}\right)\text{i}$.